Numerical Marine Hydrodynamics

- Partial Differential Equations
  - PDE Classification
  - Hyperbolic PDEs
  - Parabolic PDEs
    - Heat Equation
    - Finite Difference Schemes
      - Forward Marching (Euler)
      - Crank-Nicholson
    - Example – heat Equation
  - Elliptical PDEs

Partial Differential Equations

Quasi-linear PDE

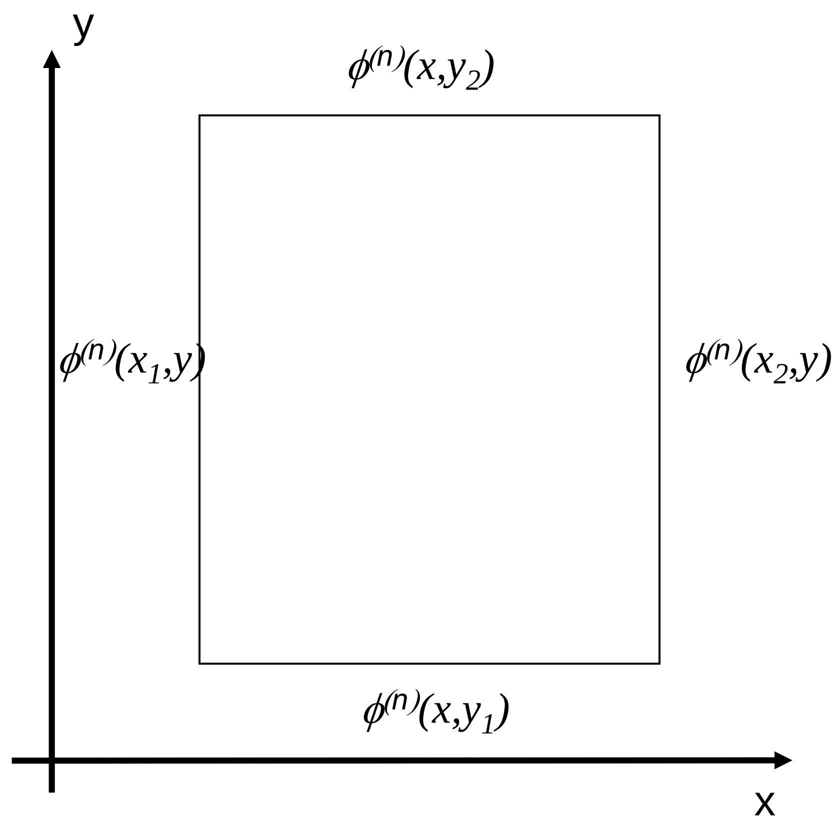
$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = F(x, y, \phi, \phi_x, \phi_y)$$

A,B and C Constants

$B^2 - 4AC > 0$ Hyperbolic

$B^2 - 4AC = 0$ Parabolic

$B^2 - 4AC < 0$ Elliptic



Partial Differential Equations
Parabolic PDE

Heat Flow Equation

$$\kappa u_{xx}(x,t) = \sigma \rho u_t(x,t), \; 0 < x < L \;, \; 0 < t < \infty$$

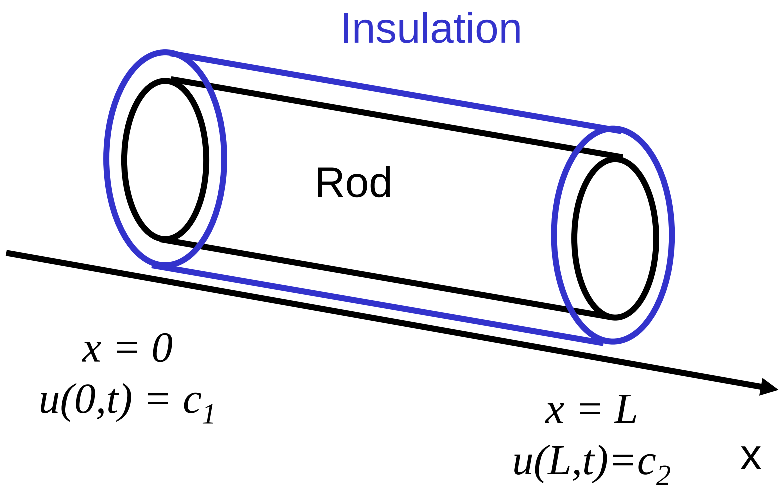
Initial Condition

$$u(x,0) = f(x), \; 0 \leq x \leq L$$

Boundary Conditions

$$\begin{aligned} u(0,t) &= c_1 \;, \; 0 < t < \infty \\ u(L,t) &= c_2 \;, \; 0 < t < \infty \end{aligned}$$

κ Thermal conductivity
σ Specific heat
ρ Density
u Temperature



IVP in one dimension, BVP in the other
Marching, Explicit or Implicit Schemes

Partial Differential Equations
Parabolic PDE

Heat Flow Equation

$$u_t(x,t) = c^2 u_{xx}(x,t),\ 0 < x < L\ ,\ 0 < t < \infty$$

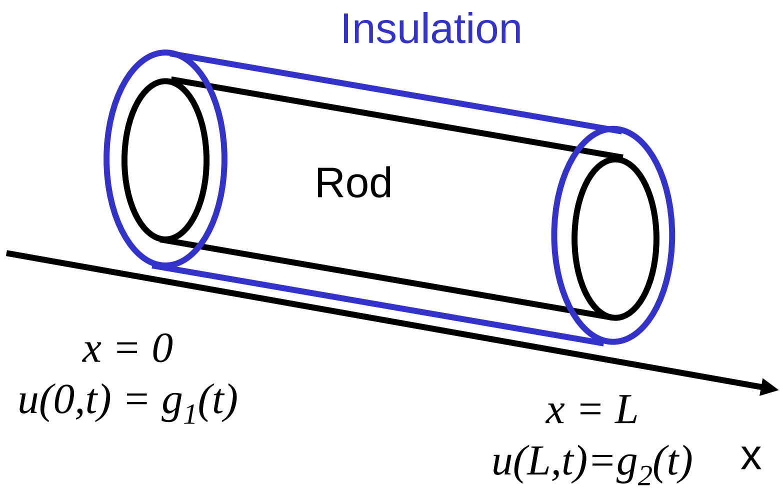
$$\sqrt{\frac{\kappa}{\rho\sigma}}$$

Initial Condition

$$u(x,0) = f(x),\ 0 \le x \le L$$

Boundary Conditions

$$\begin{aligned} u(0,t) &= g_1(t)\ ,\ 0 < t < T \\ u(L,t) &= g_2(t)\ ,\ 0 < t < T \end{aligned}$$



Partial Differential Equations
Parabolic PDE

Equidistant Sampling

$$h = L/n$$
$$k = T/m$$

Discretization

$$x_i = (i-1)h, \;\; i = 2, \ldots, n-1$$
$$t_j = (j-1)k, \;\; j = 1, \ldots, m$$

Forward Finite Difference

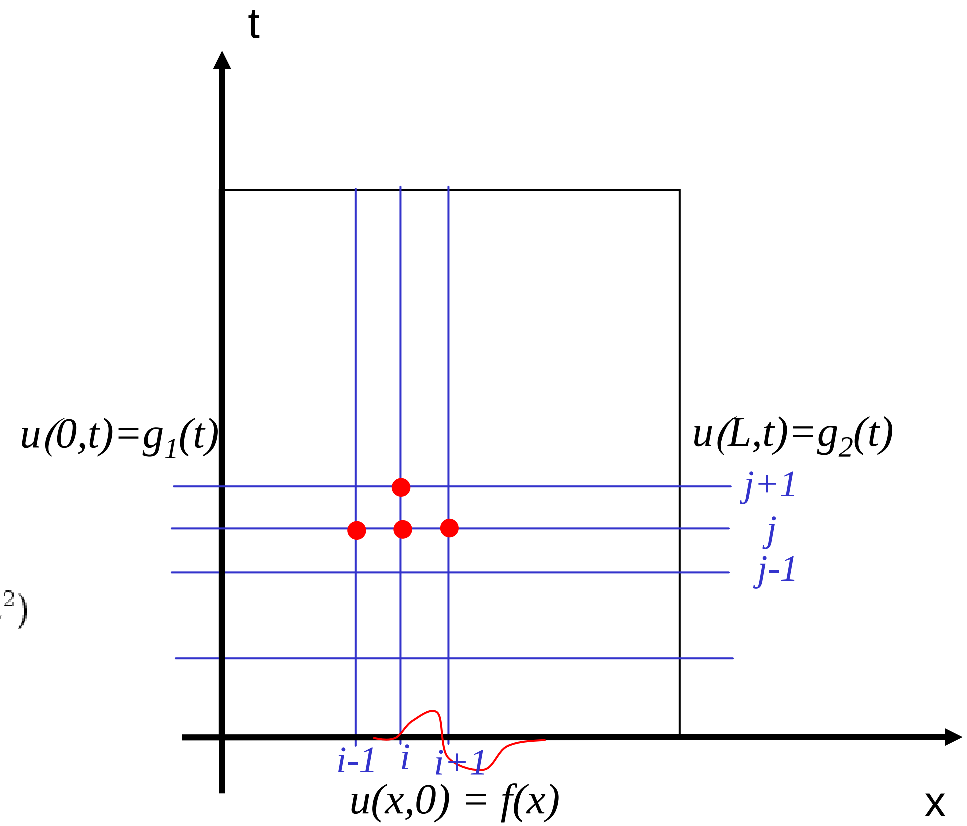
$$u_t(x,t) = \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} + O(k)$$

$$u_{xx}(x,t) = \frac{u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j)}{h^2} + O(h^2)$$

$$u_{i,j} = u(x_i, t_j)$$

Finite Difference Equation

$$\frac{u_{i,j-1} - u_{i,j+1}}{k} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$



Partial Differential Equations
Parabolic PDE

Dimensionless Flow Speed

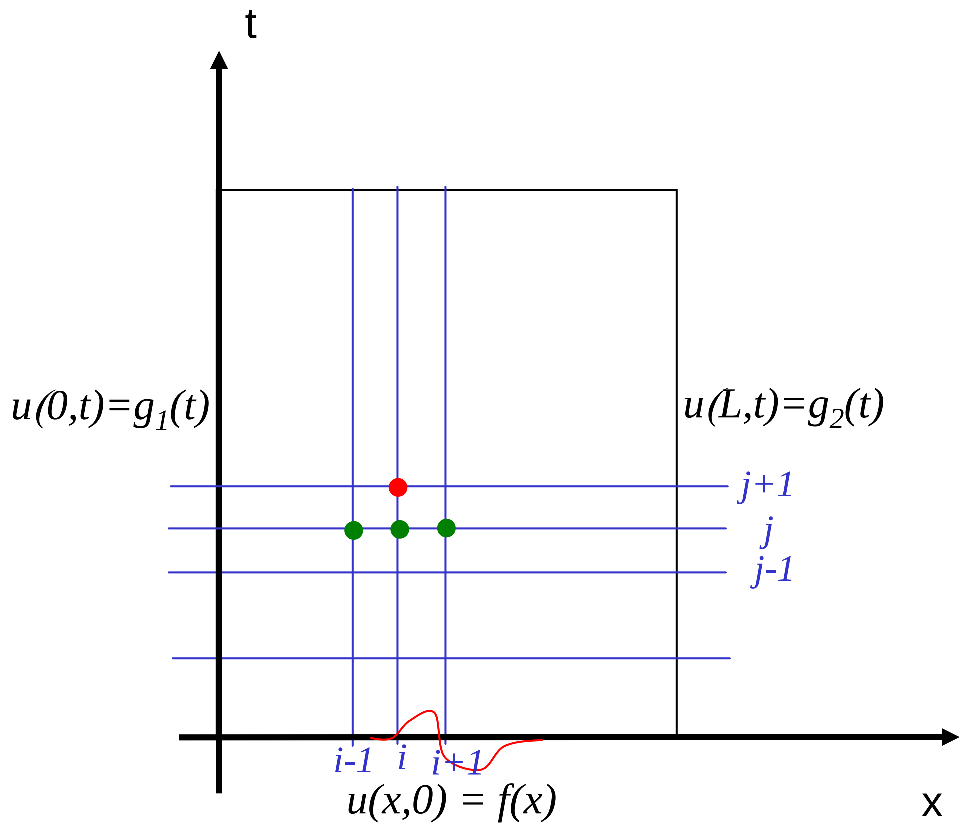
$$r = \frac{c^2 k}{h^2}$$

Explicit Finite Difference Scheme

$$u_{i,j+1} = (1-2r)u_{i,j} + r(u_{i-1,j} + u_{i+1,j})$$

Stability Requirement

$r <= 0.5$



Heat Flow Equation
Explicit Finite Differences

heat_fw.m

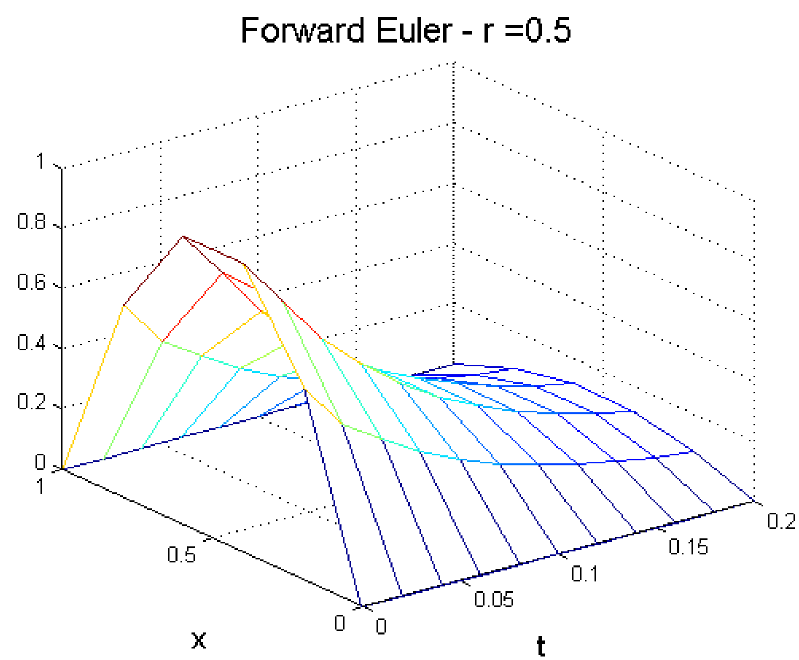
```
L=1; T=0.2; c=1;
N=5; h=L/N;
M=10; k=T/M;
r=c^2*k/h^2

x=[0:h:L]';
t=[0:k:T];
fx='4*x-4*x.^2';
g1x='0';
g2x='0';
f=inline(fx,'x');
g1=inline(g1x,'t');
g2=inline(g2x,'t');
n=length(x);
m=length(t);
u=zeros(n,m);
u(2:n-1,1)=f(x(2:n-1));
u(1,1:m)=g1(t);
u(n,1:m)=g2(t);
for j=1:m-1
    for i=2:n-1
u(i,j+1)=(1-2*r)*u(i,j) + r*(u(i+1,j)+u(i-1,j));
    end
end
 .
figure(4)
mesh(t,x,u);
a=ylabel('x');
set(a,'Fontsize',14);
a=xlabel('t');
set(a,'Fontsize',14);
a=title(['Forward Euler - r =' num2str(r)]);
set(a,'Fontsize',16);
```

$$u_t(x,t) = u_{xx}(x,t)\ ,\quad 0 < x < 1\ ,\quad 0 < t < 0.0$$

$$u(x,0) = f(x) = 4x - 4x^2$$

$$\begin{aligned} u(0,t) &= g_1(t) \equiv 0 \\ u(1,t) &= g_2(t) \equiv 0 \end{aligned}$$



Heat Flow Equation
Explicit Finite Differences

heat_fw_2.m

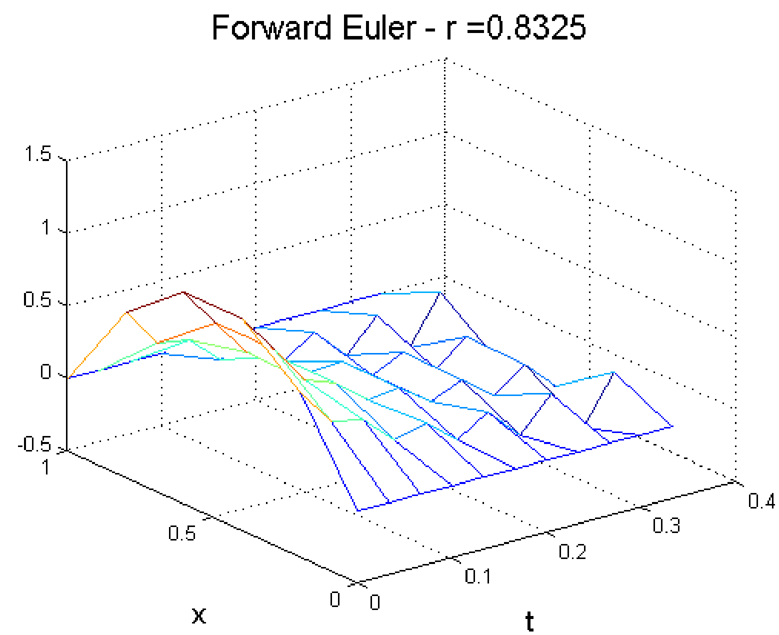
```
L=1; T=0.333; c=1;
N=5; h=L/N;
M=10; k=T/M;
r=c^2*k/h^2

x=[0:h:L]';
t=[0:k:T];
fx='4*x-4*x.^2';
g1x='0';
g2x='0';
f=inline(fx,'x');
g1=inline(g1x,'t');
g2=inline(g2x,'t');
n=length(x);
m=length(t);
u=zeros(n,m);
u(2:n-1,1)=f(x(2:n-1));
u(1,1:m)=g1(t);
u(n,1:m)=g2(t);
for j=1:m-1
    for i=2:n-1
u(i,j+1)=(1-2*r)*u(i,j) + r*(u(i+1,j)+u(i-1,j));
    end
end
 .
figure(4)
mesh(t,x,u);
a=ylabel('x');
set(a,'Fontsize',14);
a=xlabel('t');
set(a,'Fontsize',14);
a=title(['Forward Euler - r =' num2str(r)]);
set(a,'Fontsize',16);
```

$$u_t(x,t) = u_{xx}(x,t)\ ,\quad 0 < x < 1\ ,\quad 0 < t < 0.0$$

$$u(x,0) = f(x) = 4x - 4x^2$$

$$\begin{aligned} u(0,t) &= g_1(t) \equiv 0 \\ u(1,t) &= g_2(t) \equiv 0 \end{aligned}$$



Parabolic PDE
Crank-Nicholson Scheme

Equidistant Sampling

$$h = L/n$$
$$k = T/m$$

Discretization

$$x_i = (i-1)h, \; i = 2, \ldots, n-1$$
$$t_j = (j-1)k, \; j = 1, \ldots, m$$

Mid-point Finite Differences

$$u_t\left(x, t+\frac{k}{2}\right) = \frac{u(x,t+k) - u(x,t)}{k} + O(k^2)$$

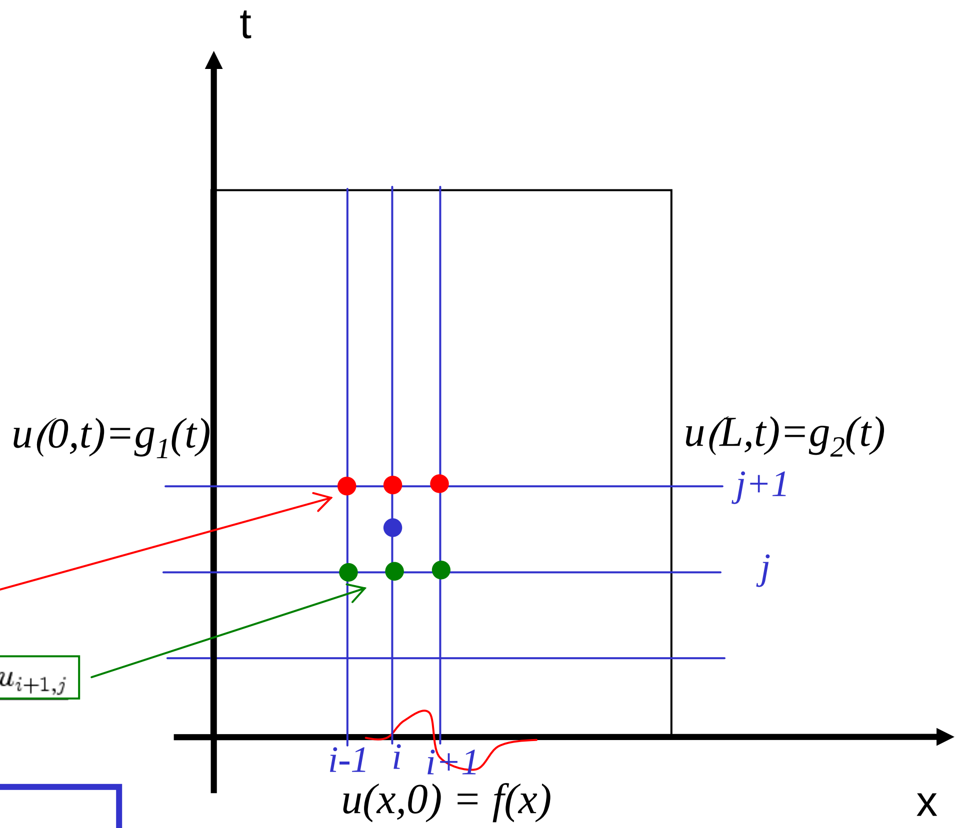
Finite Difference Equation

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c^2 \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1} + u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{2h^2}$$

Crank-Nicholson Implicit Scheme

$$r = \frac{c^2 k}{h^2}$$

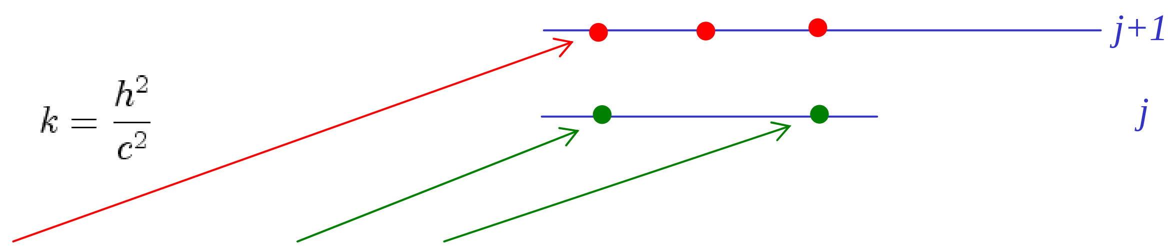
$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = (2-2r)u_{i,j} + r(u_{i-1,j} + u_{i+1,j})$$



Parabolic PDEs
Crank-Nicholson – r = 1

$$r = \frac{c^2 k}{h^2} = 1$$

$$k = \frac{h^2}{c^2}$$



$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$$

$$\begin{bmatrix} 4 & -1 & & & & & \\ -1 & 4 & -1 & & & 0 & \\ & & \cdot & \cdot & & & \\ & & -1 & 4 & -1 & & \\ & & & \cdot & \cdot & & \\ & 0 & & -1 & 4 & -1 & \\ & & & & & -1 & 4 \end{bmatrix} \begin{Bmatrix} u_{2,j+1} \\ u_{3,j+1} \\ \cdot \\ u_{i,j+1} \\ \cdot \\ u_{n-2,j+1} \\ u_{n-1,j+1} \end{Bmatrix} = \begin{Bmatrix} g_{1,j} + u_{3,j} + g_{1,j+1} \\ u_{2,j} + u_{4,j} \\ \cdot \\ u_{i-1,j} + u_{i+1,j} \\ \cdot \\ u_{n-3,j} + u_{n-1,j} \\ u_{n-2,j} + g_{n,j} + g_{n,j+1} \end{Bmatrix}$$

Heat Flow Equation
Implicit Crank-Nicholson Scheme

heat_cn.m

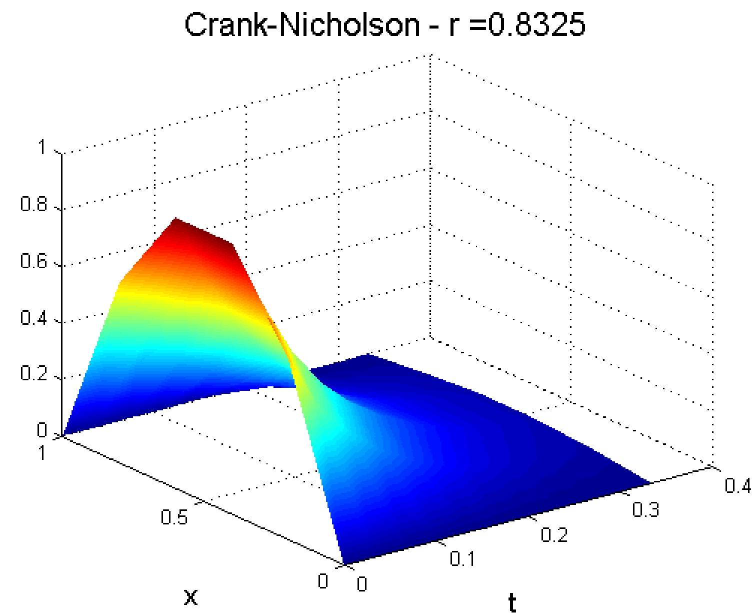
```
L=1; T=0.333; c=1;
N=5; h=L/N;
M=10;
k=T/M;
r=c^2*k/h^2

x=[0:h:L]';
t=[0:k:T];
fx='4*x-4*x.^2';
g1x='0';
g2x='0';
f=inline(fx,'x');
g1=inline(g1x,'t');
g2=inline(g2x,'t');
n=length(x); m=length(t); u=zeros(n,m);
u(2:n-1,1)=f(x(2:n-1));
u(1,1:m)=g1(t); u(n,1:m)=g2(t);
% set up Crank-Nicholson coef matrix
d=(2+2*r)*ones(n-2,1);
b=-r*ones(n-2,1);
c=b;
% LU factorization
[alf,bet]=lu_tri(d,b,c);
for j=1:m-1
    rhs=r*(u(1:n-2,j)+u(3:n,j)) +(2-2*r)*u(2:n-1,j);
    rhs(1) = rhs(1)+r*u(1,j+1);
    rhs(n-2)=rhs(n-2)+r*u(n,j+1);
% Forward substitution
    z=forw_tri(rhs,bet);
% Back substitution
    y_b=back_tri(z,alf,c);
    for i=2:n-1
      u(i,j+1)=y_b(i-1);
    end
end
```

$$u_t(x,t) = u_{xx}(x,t)\ ,\quad 0 < x < 1\ ,\quad 0 < t < 0.0$$

$$u(x,0) = f(x) = 4x - 4x^2$$

$$\begin{aligned} u(0,t) &= g_1(t) \equiv 0 \\ u(1,t) &= g_2(t) \equiv 0 \end{aligned}$$



Heat Flow Equation
Implicit Crank-Nicholson Scheme

heat_cn_sin.m

```
L=1; T=0.1; c=1;
N=10; h=L/N;
M=10;
k=T/M;
r=c^2*k/h^2

x=[0:h:L]';
t=[0:k:T];
fx='sin(pi*x)+sin(3*pi*x)';
g1x='0';
g2x='0';
f=inline(fx,'x');
g1=inline(g1x,'t');
g2=inline(g2x,'t');
n=length(x); m=length(t); u=zeros(n,m);
u(2:n-1,1)=f(x(2:n-1));
u(1,1:m)=g1(t); u(n,1:m)=g2(t);
% set up Crank-Nicholson coef matrix
d=(2+2*r)*ones(n-2,1);
b=-r*ones(n-2,1);
c=b;
% LU factorization
[alf,bet]=lu_tri(d,b,c);
for j=1:m-1
    rhs=r*(u(1:n-2,j)+u(3:n,j)) +(2-2*r)*u(2:n-1,j);
    rhs(1) = rhs(1)+r*u(1,j+1);
    rhs(n-2)=rhs(n-2)+r*u(n,j+1);
% Forward substitution
    z=forw_tri(rhs,bet);
% Back substitution
    y_b=back_tri(z,alf,c);
    for i=2:n-1
      u(i,j+1)=y_b(i-1);
    end
end
```

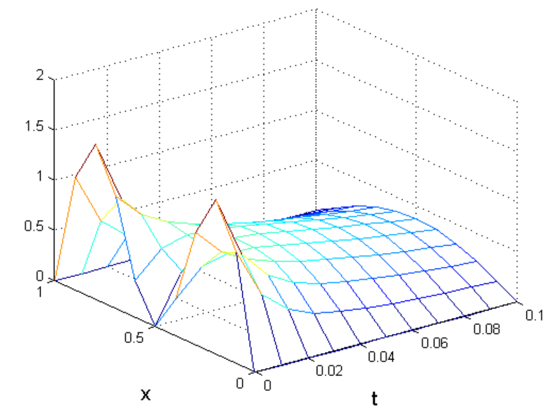
Initial Condition

$$f(x) = \sin \pi x + \sin 3\pi x$$

Analytical Solution

$$u(x,t) = e^{-\pi^2 t} \sin \pi x + e^{-9\pi^2 t} \sin 3\pi x$$

Crank-Nicholson - r =1



Crank-Nicholson Error, r = 1