Boundary Value Problems
Finite Difference Methods

Boundary Conditions with Derivatives

$$y'' - yx = g(x)$$
$$y(a) = 0$$

Central Difference

$$y'(b) = 0$$

Difference Equations

$$y_0 = 0$$
$$y_{n-1} - 2y_n + y_{n+1} - h^2 y_n x_n = h^2 g(x_n)\ , \quad n = 1,$$
$$y_N = ?$$

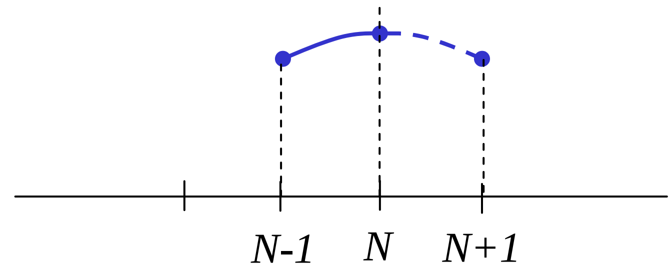
Backward Difference

$$y'(b) = 0 = \frac{y_N - y_{N-1}}{h} + O(h)$$

$$y_0 = 0$$
$$y_{n-1} - 2y_n + y_{n+1} - h^2 y_n x_n = h^2 g(x_n)\ , \quad n = 1, 2, \ldots N-1$$
$$y_N - y_{N-1} = 0$$

O(h^4)

O(h^2)



Central Difference

$$y'(b) = 0 = \frac{y_{N+1} - y_{N-1}}{2h} + O(h^2)$$

$$y_0 = 0$$
$$y_{n-1} - 2y_n + y_{n+1} - h^2 y_n x_n = h^2 g(x_n)\ , \quad n = 1, 2, \ldots N-1$$
$$2(y_{N-1} - y_N) - h^2 y_N x_N = 0$$

O(h^3)

General Boundary Conditions

$$p_0 y(b) + p_1 y'(b) = p_2$$

Finite Difference Representation

$$p_0 y_N + \frac{p_1(y_{N+1} - y_{N-1})}{2h} = p_2$$

Add extra point - N equations, N unknowns

Numerical Marine Hydrodynamics

- Partial Differential Equations
  - PDE Classification
  - Hyperbolic PDEs
    - Finite Difference Solutions
    - Wave Equation
      - D'Alambert's Principle
      - Method of Characteristics
  - Parabolic PDEs
  - Elliptical PDEs

Partial Differential Equations

Quasi-linear PDE

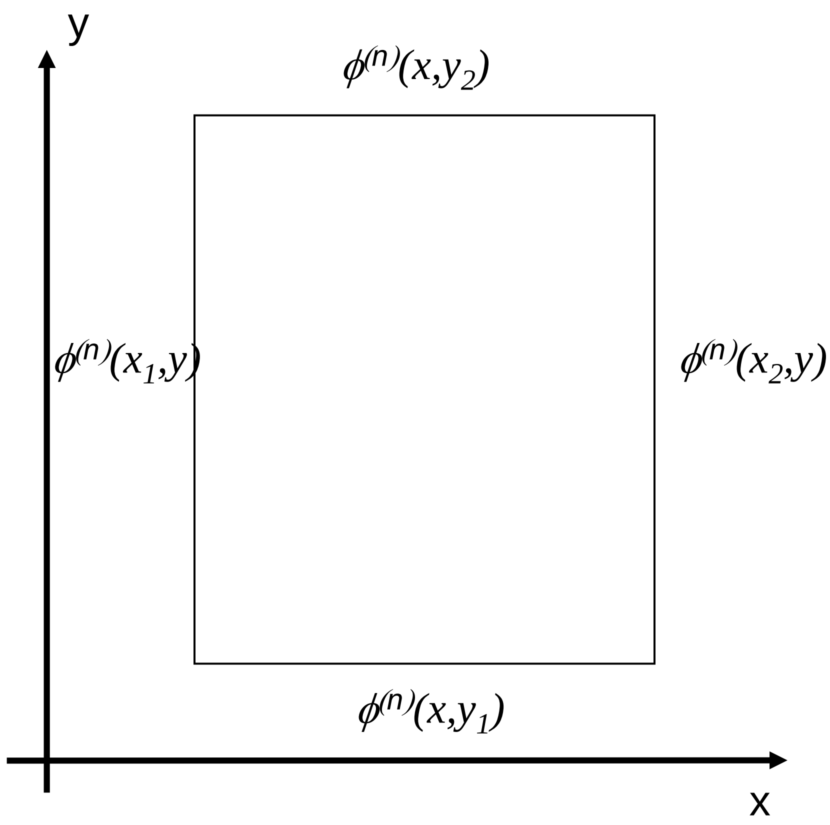
$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = F(x, y, \phi, \phi_x, \phi_y)$$

A,B and C Constants

$$B^2 - 4AC > 0$$ Hyperbolic

$$B^2 - 4AC = 0$$ Parabolic

$$B^2 - 4AC < 0$$ Elliptic



Partial Differential Equations
Hyperbolic PDE

Waves on a String

$$\rho u^{tt}(x,t) = T u_{xx}(x,t), \;\; 0 < x < L \;, \; 0 < t < \infty$$

Initial Conditions

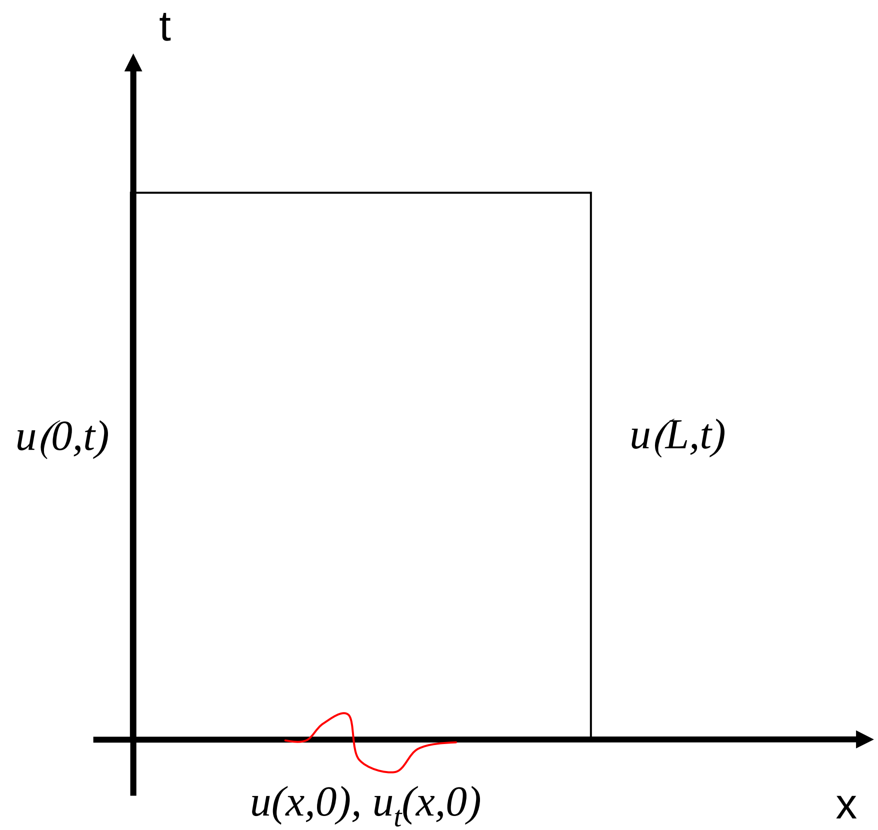
$$\begin{aligned} u(x,0) &= f(x), \;\; 0 \le x \le L \\ u_t(x,0) &= g(x), \;\; 0 < x < L \end{aligned}$$

Boundary Conditions

$$\begin{aligned} u(0,t) &= 0, \;, \; 0 < t < \infty \\ u(L,t) &= 0, \;, \; 0 < t < \infty \end{aligned}$$

Wave Solutions

$$u = \begin{cases} F(x-ct) & \text{Forward propagating wave} \\ G(x+ct) & \text{Backward propagating wave} \end{cases}$$



Typically Initial Value Problems in Time, Boundary Value Problems in Space
Time-Marching Solutions – Explicit Schemes Generally Stable

Partial Differential Equations
Parabolic PDE

Heat Flow Equation

$$\kappa u_{xx}(x,t) = \sigma \rho u_t(x,t),\ 0 < x < L\ ,\ 0 < t < \infty$$

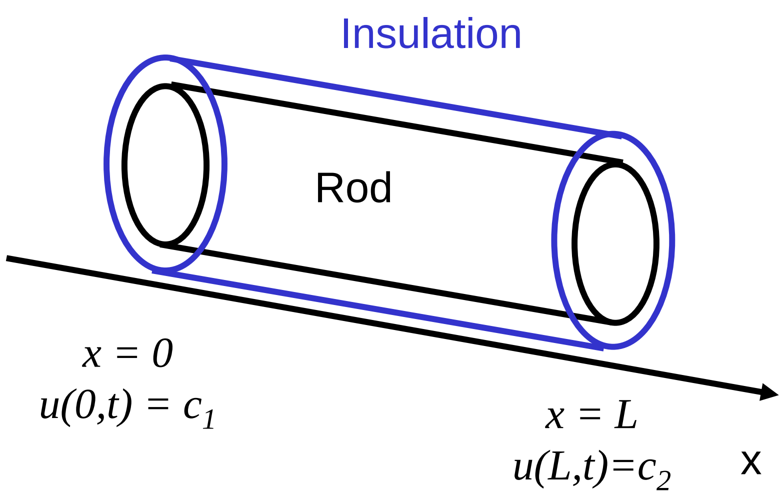
Initial Condition

$$u(x,0) = f(x),\ 0 \le x \le L$$

Boundary Conditions

$$u(0,t) = c_1\ ,\ 0 < t < \infty$$
$$u(L,t) = c_2\ ,\ 0 < t < \infty$$

κ Thermal conductivity
σ Specific heat
ρ Density
u Temperature



IVP in one dimension, BVP in the other
Marching, Explicit or Implicit Schemes

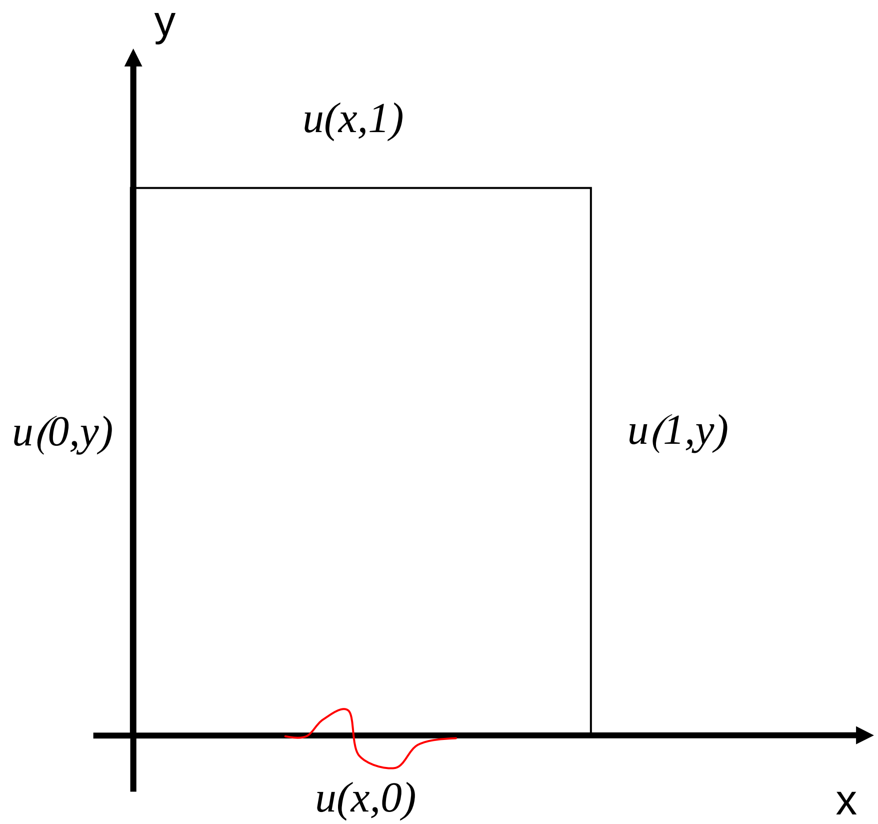
Partial Differential Equations
Elliptic PDE

Potential Flow in a Duct
Laplace Equation

$$u_{xx} + u_{yy} = -v_0$$

Boundary Conditions

$$\begin{aligned} u(x,0) &= f_1(x) \\ u(x,1) &= f_2(x) \\ u(0,y) &= f_3(y) \\ u(1,y) &= f_4(y) \end{aligned}$$



BVP in both Dimensions
Global Finite Difference Solution

Partial Differential Equations
Hyperbolic PDE

Wave Equation

$$\rho u^{tt}(x,t) = T u_{xx}(x,t), \quad 0 < x < L \;,\; 0 < t < \infty$$

Discretization

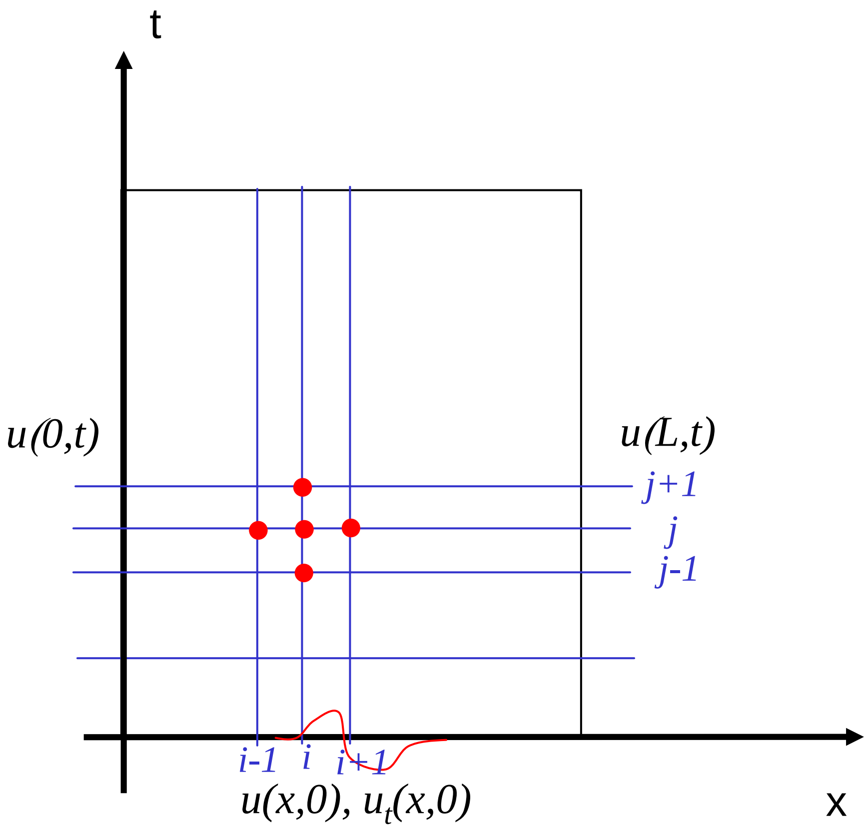
$$\begin{aligned} h &= L/n \\ k &= T/m \\ x_i &= (i-1)h, \; i = 2, \ldots, n-1 \\ t_j &= (j-1)k, \; j = 1, \ldots, m \end{aligned}$$

Finite Difference Representations

$$u_{tt}(x,t) = \frac{u(x_i, t_{j-1}) - 2u(x_i, t_j) + u(x_i, t_{j+1})}{k^2} + O(k^2)$$

$$u_{xx}(x,t) = \frac{u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j)}{h^2} + O(h^2)$$

$$u_{i,j} = u(x_i, t_j)$$



Finite Difference Representations

$$\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Partial Differential Equations
Hyperbolic PDE

Dimensionless Wave Speed

$$C = \frac{ck}{h}$$

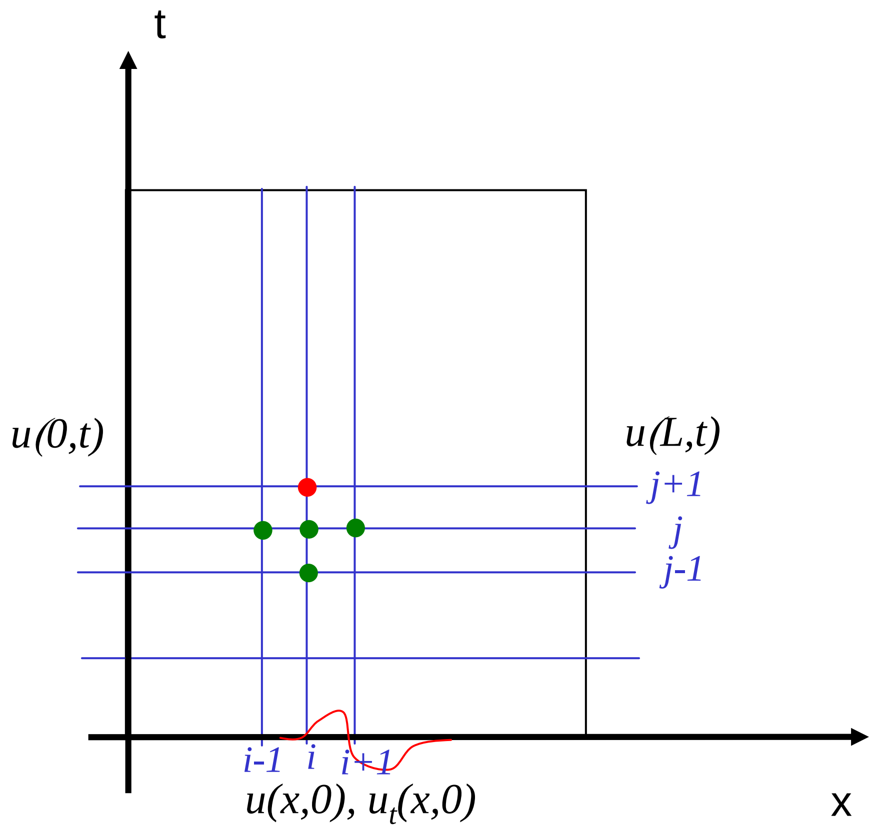
$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = C^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

Explicit Finite Difference Scheme

$$u_{i,j+1} = (2 - 2C^2)u_{i,j} + C^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}, \quad i = 2, \ldots n-1$$

Stability Requirement

$$C = \frac{ck}{h} < 1$$



Partial Differential Equations
Hyperbolic PDE

Euler Starter

$$u_{i,2} = u(x_i, k) \simeq u(x_i, 0) + k u_t(x, 0) k = \boxed{f(x_i) + k g(x_i)}$$

Second Derivative Known

$$u_{xx}(x, 0) = f''$$

From Wave Equation

$$u_{tt}(x_i, 0) = c^2 u_{xx}(x_i, 0) = c^2 f_{xx}(x_i) = c^2 \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} + O(h^2)$$

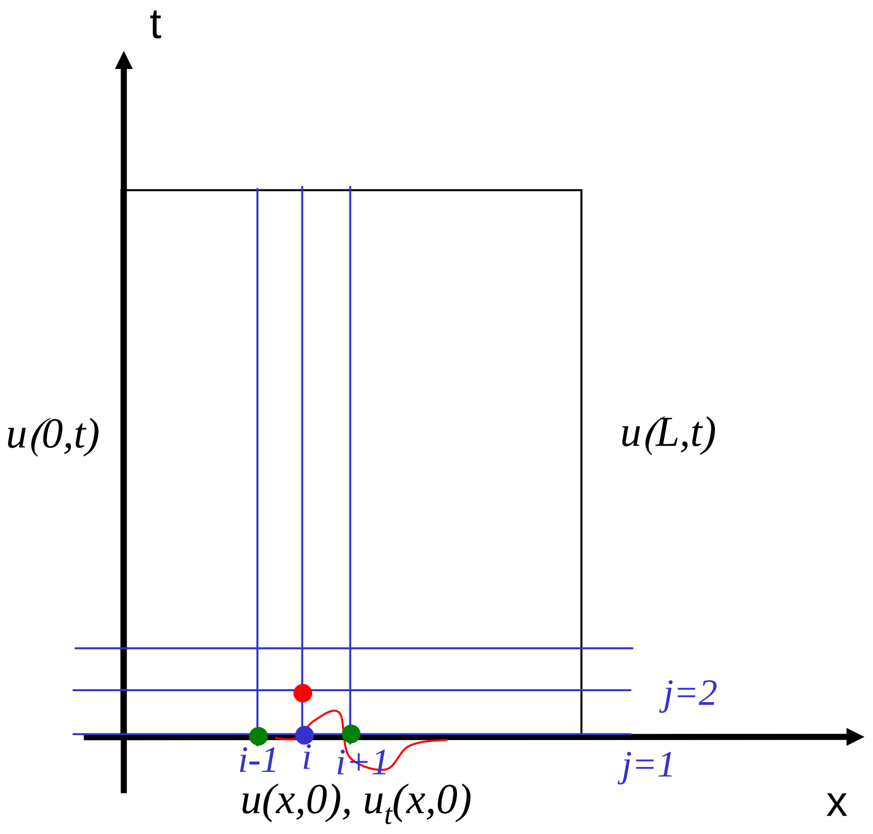
Taylor Expansion

$$u(x,k) = u(x,0) + u_t(x,0) + \frac{u_{tt}(x,0)k^2}{2} + O(k^3)$$

Higher Order Self Starter

$$\boxed{u_{i,2} = u(x_i, k)} = f_i + k g_i + \frac{c^2 k^2}{2h^2}(f_{i-1} - f_i + f_{i+1}) + O(h^2 k^2) + O(k^3)$$

$$= \boxed{\left(1 - C^2\right) f_i + k g_i} + \boxed{\frac{C^2}{2}(f_{i+1} + f_{i-1})}$$



Wave Equation
d'Alembert's Solution

Wave Equation

$$\rho u^{tt}(x,t) = T u_{xx}(x,t), \;\; 0 < x < L \;, \; 0 < t < \infty$$

Solution

$$u(x,t) = F(x-ct) + G(x+ct) \;, \; 0 < x < L$$

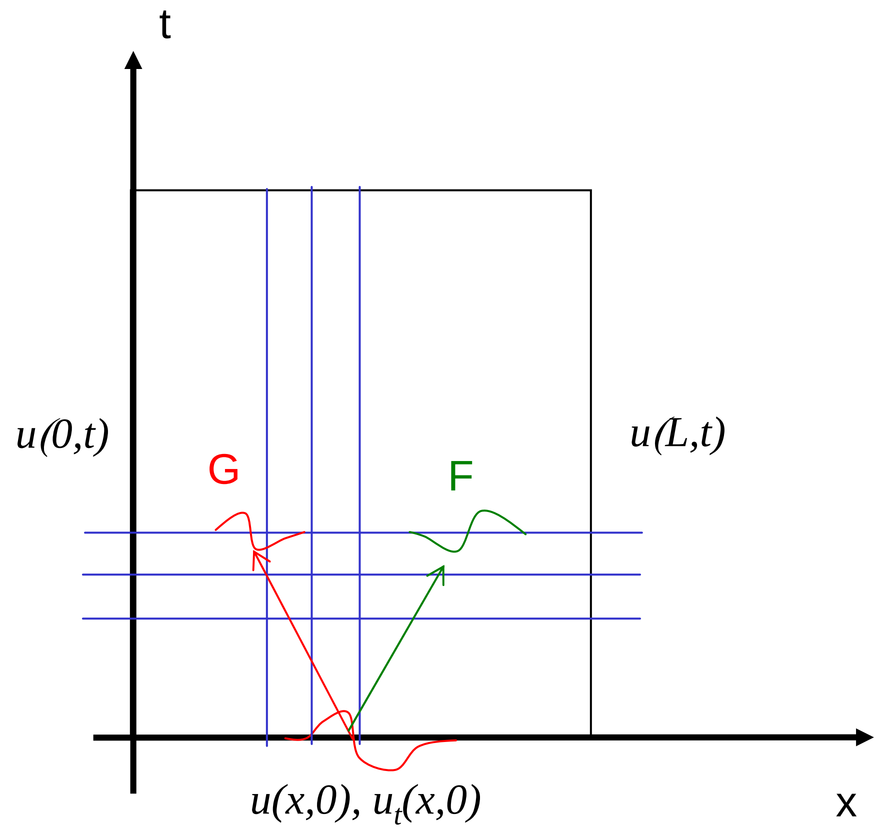
Periodicity

$$\begin{aligned} F(-z) &= -F(z) \\ f(z+2L) &= F(z) \end{aligned}$$

$$\begin{aligned} G(-z) &= -G(z) \\ G(z+2L) &= G(z) \end{aligned}$$

Proof

$$\begin{aligned} u_{xx}(x,t) &= F''(x-ct) + G''(x+ct) \\ u_{tt}(x,t) &= c^2 F''(x-ct) + c^2 G''(x+ct) \\ &= c^2 u_{xx}(x,t) \end{aligned}$$



Hyperbolic PDE
Method of Characteristics

Explicit Finite Difference Scheme

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = C^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

$$u_{i,j+1} = (2 - 2C^2)u_{i,j} + C^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}, \;\; i = 2, \ldots n-1$$

First 2 Rows known

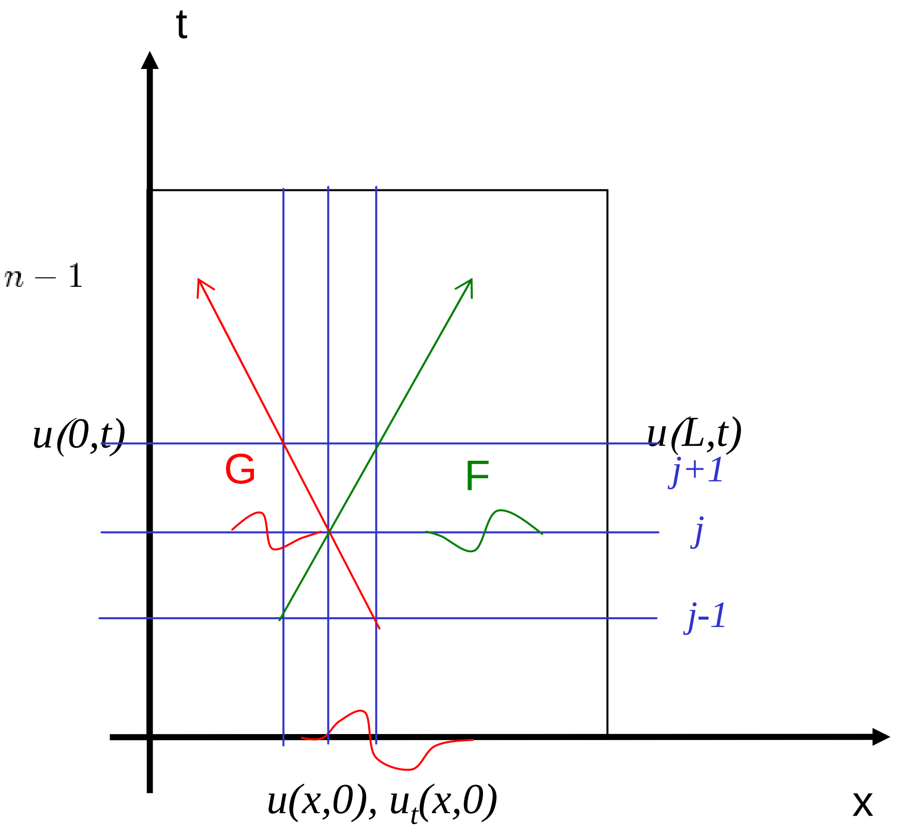
$$u_{i,1} = u(x_i, 0)$$
$$u_{i,2} = u(x_i, k)$$

Characteristic Sampling

$$k = h/c \Rightarrow C = 1$$

Exact Discrete Solution

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$



Hyperbolic PDE
Method of Characteristics

Exact Discrete Solution

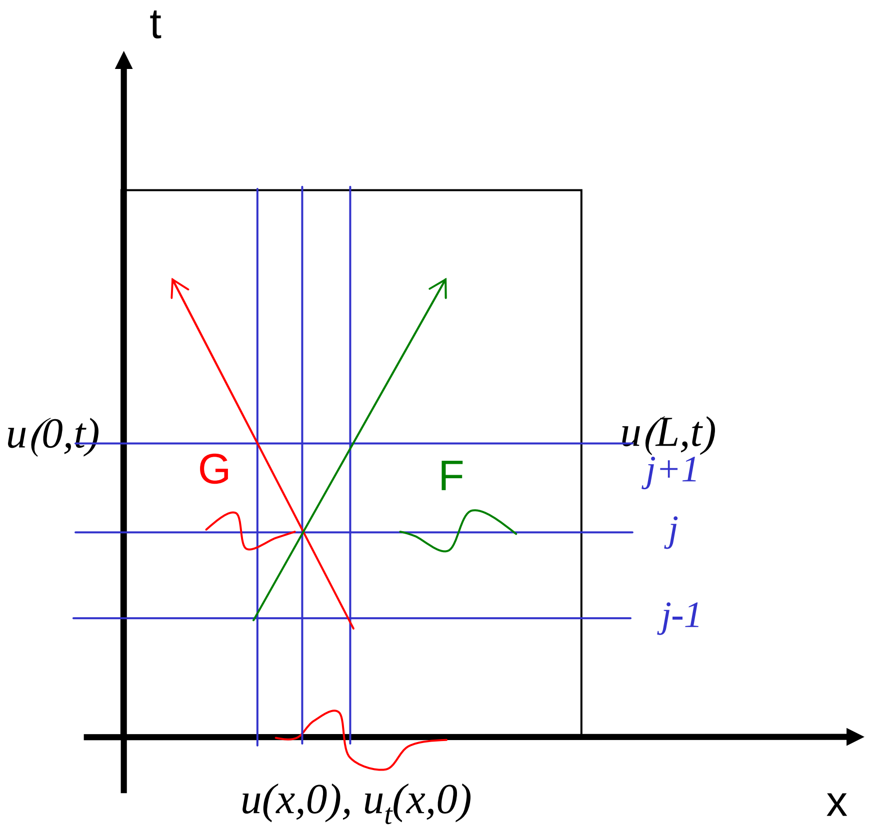
$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$

D'Alembert's Solution

$$\begin{aligned} x_i - ct_j &= (i-1)h - c(j-1)k \\ &= (i-1)h - (j-1)h \\ &= (i-j)h \end{aligned}$$

$$\begin{aligned} x_i + ct_j &= (i-1)h + c(j-1)k \\ &= (i-1)h + (j-1)h \\ &= (i+j-2)h \end{aligned}$$

$$u_{i,j} = F((i-j)h) + G((i+j-2)h)$$



Proof

$$\begin{aligned} & u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \\ = \; & F((i+1-j)h) + F((i-1-j)h) - F((i-(j-1))h) \\ & +G((i+1+j-2)h) + G((i-1+j-2)h) - G((i+j-1-2)h) \\ = \; & F((i-(j+1))h) + G((i+(j+1)-2)h) \\ = \; & u_{i,j+1} \end{aligned}$$

Waves on a String

waveeq.m

```
L=10;
T=10;
c=1.5;
N=100;
h=L/N;
M=400;
k=T/M;
C=c*k/h
Lf=0.5;
x=[0:h:L]';
t=[0:k:T];
%fx=['exp(-0.5*(' num2str(L/2) '-x).^2/(' num2str(Lf) ').^2)'];
%gx='0';
fx='exp(-0.5*(5-x).^2/0.5^2).*cos((x-5)*pi)';
gx='0';
f=inline(fx,'x');
g=inline(gx,'x');

n=length(x);
m=length(t);
u=zeros(n,m);
u(2:n-1,1)=f(x(2:n-1));
for i=2:n-1
 u(i,2) = (1-C^2)*u(i,1) + k*g(x(i)) +C^2*(u(i-1,1)+u(i+1,1))/2;
end

for j=2:m-1
   for i=2:n-1
u(i,j+1)=(2-2*C^2)*u(i,j) + C^2*(u(i+1,j)+u(i-1,j)) - u(i,j-1);
   end
end
```