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2.29 NUMERICAL FLUID MECHANICS— SPRING 2007

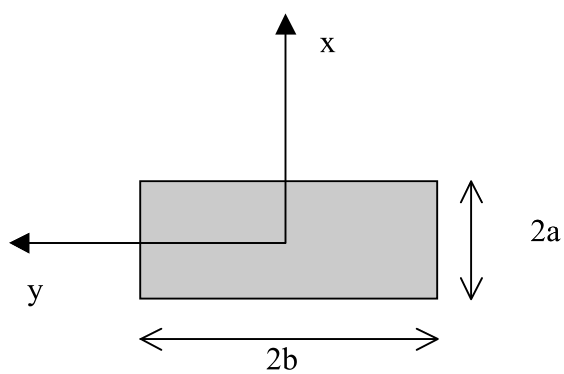
Solution of Problem Set 4

Totally 120 points
Posted 04/21/07, due Thursday 4 p.m. 05/08/07, Focused on Lecture 18 to 22

Problem 4.1 (120 points): Steady state unidirectional fully developed incompressible laminar flow in a rectangular pipe

a) Write down the Navier-Stokes equation for the fully developed incompressible laminar flow in a pipe with arbitrary constant cross section and simplify the equations[1]. Furthermore simplify the equation for the case when the pressure gradient is fixed. Explain when your assumption holds.

b) What is the name of this equation? Categorize that and give at least five examples of other cases (possibly from other physical domains) where we encounter the same equation.



Now consider the upper rectangular pipe where $a \le b$.

c) Find an analytical solution by separation of variables.

d) (EXTRA CREDIT 10 Points) Explain how your solution (method) changes if we had a pulsating flow. You do not need to solve it thoroughly.

[1] If you have difficulty in deriving the equations you can look at different basic textbooks. In particular "Analysis of Transport Phenomena" by W. M. Deen can be suggested.

e) Compute the maximum shear stress and the net volumetric flow rate.

f) Compare the maximum shear stress and the pressure gradient in a square pipe with a circular pipe with the same area and the same flow rate.

By now we have solved the equation analytically for our simple cross section, but in general it will be very hard or almost impossible to solve it analytically for arbitrary cross sections. Indeed we usually have to rely on numerical methods and here we will develop a finite difference method to solve the equation.

g) Develop a finite difference scheme to solve the equation for the square pipe. Start by a mesh consisting of 2 elements (3 nodes) in each directions and refine it in each step by a factor of two until your net flow rate computed has a relative error of less than 0.1% compared to the previous step. Use the proper integration for computing the flow rate.

h) Compare the flow rate computed above with the analytical solution.

i) Plot the flow rate as a function of mesh size and discuss the curve slope.

j) Plot the numerical and analytical velocity contours.

k) Due to Laplacian operator in the equations it sounds appealing to use a uniform grid. However, it is not always possible to use a uniform grid, especially in extreme cases like when $\frac{a}{b} \to 0$. To manipulate those cases it is very good to nondimensionalize the equation. So nondimensionalize the equation with $\eta = \frac{x}{a}, \xi = \frac{y}{b}$.

l) Solve the finite difference equation of previous part in (η, ξ) domain for the case where "b=10a". Use the same number of elements as the last step of part g.

m) Compute the flow rate of previous part and compare it to the analytical solution. Also plot the velocity contours as well as shear stress contours.

n) Use the MATLAB PDE tool and solve the equation of part k with a mesh preferably as much refined as part "l". Explain clearly how you compute the volumetric flow rate and repeat part "m".

o) (EXTRA CREDIT 10 Points) Now consider the case where $\frac{a}{b} \to 0$. Compute the volumetric flow rate normalized by "b" (flow rate per unit of depth) from analytical solution. Compare it with similar 1D problem and discuss whether they are equivalent or not. Also discuss what happens in the numerical solution and how it changes.

Solution:

a)

The axis of channel is described by "z" coordinate. In general we have $\vec{V} = \vec{V}(x, y, z, t)$. However:

- Laminar Flow: $V_x = V_y = 0$. We are assuming that we have a unidirectional flow, and this is true provided that there is no rotational (swirling) component of flow introduced from upstream. This is also consistent with assuming a constant cross section for the pipe provided that we have a constant curvature and torsion (which holds for fully developed flow).
- Steady Flow: $\vec{V} = \vec{V}(x, y, z)$
- Fully developed flow: $\vec{V} = \vec{V}(x, y, t)$
- Ignore gravity effect or include it into hydrostatic pressure. From here forward we only deal with dynamic pressure.

The above assumptions lead to $V_x = V_y = 0,\ V_z = V_z(x, y)$. So we can use the Navier-Stokes equation:

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}.\nabla\vec{V}\right) = -\nabla P + \mu\nabla^2\vec{V}$$

However, to proceed from here we ADDITIOANLLY assume that the channel axis is straight (otherwise think of e.g. fully developed steady state flow in a pipe with constant curvature and torsion). As a result of this extra assumption we can apply Navier-Stokes equation in Cartesian coordinate. For example the "x" coordinate equation will be:

$$\rho\left(\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right)$$

Insert velocity into Navier-Stokes (while initially ignoring steady flow assumption):

$$\text{x axis: } 0 = -\frac{\partial P}{\partial x}$$

$$\text{y axis: } 0 = -\frac{\partial P}{\partial y}$$

$$\text{z axis: } 0 = \underbrace{-\frac{\partial P}{\partial z}}_{\text{function of z,t}} \underbrace{-\rho\frac{\partial V_z}{\partial t} + \mu\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2}\right)}_{\text{function of x,y,t}}$$

The first two equations lead to $P = P(z, t)$. However, from the 3[rd] equation we can see that $\frac{\partial P}{\partial z}$ can only be a function of time (not "z"). Furthermore, if we incorporate the steady flow assumption we will have:

$$\frac{\partial P}{\partial z} = \text{constant} = \frac{dP}{dz} =$$
$$\mu(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2}) = \frac{dP}{dz}$$

So the equation can be represented as:

$$\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} = \frac{1}{\mu}\frac{dP}{dz}$$
$$\nabla^2 V_z(x,y) = \frac{1}{\mu}\frac{dP}{dz} = \text{constant}$$

b)

This is a special case of Poisson's equation with constant source in 2D domain. It can be also categorized as a 2D elliptic partial differential equation with homogeneous Dirichlet boundary condition. Some other cases where we have the same equation include[2]:

- Electrostatic potentials: $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$, where "V" is the electrostatic potential, ρ is the charge density and ε_0 is the electrical permittivity.
- Gravitational potentials: $\nabla^2 V = 4\pi G\rho$, where "V" is the gravitational potential, ρ is the mass density and "G" is the gravitational constant.
- Heat transfer: $\nabla^2 T = -\frac{H_v}{k}$, where "T" is the temperature, "k" it thermal conductivity and "H_v" is the rate of energy input per unit volume from external power sources (for example from a reaction).
- Diffusion: $\nabla^2 C = -\frac{R_v}{D}$, where "C" is total molar concentration of a species, "D" it the diffusion coefficient of that and "R_v" is the rate of formation of that species per unit volume (for example from a reaction).
- Transverse deflection of elastic membrane: $\nabla^2 z(x,y) = -\frac{p}{N}$, where "z" is the deflection, "p" is pressure load and "N" is the constant tension force per unit length.

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[2] Other examples can be found at: "Introduction to the Finite Element Method", J.N. Reddy, McGraw Hill Publishers, 2nd Edition, Page 7 or http://math.nist.gov/mcsd/savg/tutorial/ansys/FEM/.

- Torsion of arbitrary cross sections: $\nabla^2\phi(x,y) = -2G\theta$, where "$\phi$" is the stress function ($\tau_{zx} = \frac{\partial\phi}{\partial x}$, $\tau_{zy} = \frac{\partial\phi}{\partial y}$), "G" is shear modulus load and "θ" is the angle of twist per unit length.
- Transverse deflection of plate: $\nabla^4 z(x,y) = \frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = \frac{q}{D}$, where "z" is the deflection, "q" is pressure load and "D" is a constant called flexural rigidity of the plate (related to plate thickness, modulus of elasticity and Poisson's ratio). This equation can be broken to two Poisson's equations: $\nabla^2 z(x,y) = u,\ \nabla^2 u(x,y) = \frac{q}{D}$.
- Pressure Poisson's equation: $\nabla^2 P = f(\vec{V}, \nu)$, where "P" is the pressure, "V" is the velocity and "ν" is the kinematics viscosity. This equation is derived by taking the divergence of the momentum equation (only applicable for an incompressible flow field).

c)

$$\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} = \frac{1}{\mu}\frac{dP}{dz}$$

We define a new variable $u = \frac{V_z}{\frac{1}{2\mu}\frac{dP}{dz}}$. So we have the new equation:

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2$$

The BC (boundary condition) for the above equation comes from the zero velocity of fluid particle at the pipe wall:

$$u(x = \pm a, y \qquad) = 0$$
$$u(x \qquad, y = \pm b) = 0$$

While the BCs are homogenous (equal to zero), the Poisson equation itself is not homogenous. So we need to homogenize it (transform it to the Laplace equation) to be able to utilize the method of separation of variables. Since the equation and the BCs are linear we can achieve this easily by assuming $u = v + w$, where "v" is a particular solution and "w" is the solution of the Laplace equation:

$$\nabla^2 v(x,y) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 2$$

$$\nabla^2 w(x,y) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

The "v" solution should be chosen cleverly such that at least one set of boundary conditions on "w" will be homogenous (otherwise we cannot apply the method of separation of variables on "w").

Two appealing choices are $v = x^2 - a^2$ or $v = y^2 - b^2$. However, we chose $v = x^2 - a^2$ because "a<b" and this will represent the dominant term of solution for ideal cases where $\frac{a}{b} \to 0$. So now we can recalculate the BC values for "w":

$$w(x = \pm a, y \qquad) = 0$$

$$w(x \qquad, y = \pm b) = -(x^2 - a^2)$$

To solve the equation on "w" we assume $w(x,y) = X(x)Y(y)$. So we have:

$$X(\pm a) = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases}$$

To find the conditions on "λ", we start with "X" equation because that is the one whose BCs are separated:

$$X'' + \lambda X = 0, \qquad X(\pm a) = 0$$

$$\begin{cases} \lambda = -\alpha^2 < 0 \Rightarrow X = C_1 e^{\alpha x} + C_2 e^{-\alpha x} & (1) \\ \lambda = 0 \qquad \Rightarrow X = C_1 + C_2 x & (2) \\ \lambda = +\alpha^2 > 0 \Rightarrow X = C_1 \cos(\alpha x) + C_2 \sin(\alpha x) & (3) \end{cases}$$

The only nontrivial (nonzero) solution corresponds to the case when "λ" is greater than zero. To do so we have:

$$X(+a) = C_1 \cos(\alpha a) + C_2 \sin(\alpha a) = 0$$
$$X(-a) = C_1 \cos(\alpha a) - C_2 \sin(\alpha a) = 0$$

For the nontrivial solution we have:

$$\det\left(\begin{bmatrix} \cos(\alpha a) & \sin(\alpha a) \\ \cos(\alpha a) & -\sin(\alpha a) \end{bmatrix}\right) = 0$$

$$\sin(2\alpha a) = 0 \Rightarrow 2\alpha a = k\pi \qquad k = 1,2,3,....$$

This leads to two series of solutions:

1) $\cos((n+\frac{1}{2})\pi\frac{x}{a})$ corresponding to $\alpha = (n+\frac{1}{2})\frac{\pi}{a}$, $n = 0,1,2,....$

2) $\sin(n\pi\frac{x}{a})$ corresponding to $\alpha = n\frac{\pi}{a}$, $n = 1,2,3,....$

However, due to the symmetry, only the 1st series of solution exists for this problem. As a result we have:

$$\lambda_n = \alpha_n{}^2 = \left((n+\frac{1}{2})\frac{\pi}{a}\right)^2, \quad X_n(x) = \cos(\alpha_n x)$$

The Y(y) can be obtained accordingly:

$$Y'' - \lambda Y = 0 \Rightarrow Y_n(x) = C_1 \cosh(\alpha_n x) + C_2 \sinh(\alpha_n x)$$

The symmetry on "Y" implies that C_2 is zero so:

$$\alpha_n = (n+\frac{1}{2})\frac{\pi}{a}, \quad X_n(x) = \cos(\alpha_n x),\ Y_n(y) = \cosh(\alpha_n y)$$

The "w" solution will be obtained from the weighted series summation:

$$w_n(x,y) = X_n(x)Y_n(y)$$
$$w(x,y) = \sum_{n=0}^{n=\infty} C_n w_n(x,y) = \sum_{n=0}^{n=\infty} C_n \cosh(\alpha_n y)\cos(\alpha_n x)$$

The C_n weights have to be calculated from boundary conditions:

$$w(x, y = \pm b) = -(x^2 - a^2) = \sum_{n=0}^{n=\infty} C_n \cosh(\alpha_n b)\cos(\alpha_n x)$$

The weights can be computed from Fourier expansion by $\cos(\alpha_n x)$:

$$w(x, y=\pm b) = -(x^2 - a^2) = \sum_{n=0}^{n=\infty} C_n \cosh(\alpha_n b)\cos(\alpha_n x)$$

$$C_n \cosh(\alpha_n b) = \frac{\int_{-a}^{+a} -(x^2 - a^2)\cos(\alpha_n x)dx}{\int_{-a}^{+a} \cos^2(\alpha_n x)dx} = \frac{2}{a}\int_0^{+a} -(x^2 - a^2)\cos(\alpha_n x)dx$$

$$C_n \cosh(\alpha_n b) = \frac{4}{a}\frac{\cos(n\pi)}{\alpha_n^3} = \frac{4}{a}\frac{(-1)^n}{\alpha_n^3}$$

So the solution will be in this form:

$$u(x,y) = v + w = (x^2 - a^2) + \sum_{n=0}^{n=\infty} \frac{4}{a}\frac{(-1)^n}{\alpha_n^3}\frac{\cosh(\alpha_n y)}{\cosh(\alpha_n b)}\cos(\alpha_n x)$$

d)

The equation can be rather complicated because in general even the nonlinear terms have to be included in the Navier-Stokes equation.

$$\rho\left(\frac{\partial V_z}{\partial t} + V_z \frac{\partial V_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right)$$

For low Reynolds numbers we will have:

$$\rho\left(\frac{\partial V_z}{\partial t}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right)$$

The above equation is linear and we can apply the method of separation of variables for general case of arbitrary driving pressure. Particularly, the "$\frac{\partial P}{\partial z} = \text{constant}$" part will generate the same solution as before.

However, now consider the case where $\frac{\partial P}{\partial z} = Ae^{iwt}$ [3]. Now if the pressure wave is too long (compared to channel size so that $\frac{\partial V_z}{\partial z} << \frac{\partial V_z}{\partial x}, \frac{\partial V_z}{\partial y}$ [4]), then we have:

[3] Here we are looking at a particular z. However, the velocity of other axial locations will have some phase lag with respect to this particular z (look at below note for further details).

$$\rho(\frac{\partial V_z}{\partial t}) = -\frac{\partial P}{\partial z} + \mu(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2})$$

The steady solution to this system can be easily obtained by plugging $V_z(x,y,z,t) = \tilde{V}(x,y)e^{iwt}$ into the equation so we have:

$$\rho(-iw\tilde{V}) = -A + \mu(\frac{\partial^2 \tilde{V}}{\partial x^2} + \frac{\partial^2 \tilde{V}}{\partial y^2})$$

This equation can be solved by separation of variables. However, it will be rather much more complicated due to newly added imaginary term. Consequently here $\tilde{V}(x,y)$ will be a complex number and includes some phase lag [5].

e)

The volumetric flow rate can be computed from velocity integration:

$$Q = \int_A V_z dA = \frac{1}{2\mu}\frac{dP}{dz} \times 4\int_0^a \int_0^b u dy dx$$

$$Q = \frac{2}{\mu}\frac{dP}{dz} \times \left\{ -\frac{2}{3}a^3 b + \sum_{n=0}^{n=\infty} \frac{4}{a}\frac{(-1)^n}{\alpha_n^5}\frac{\sinh(\alpha_n b)}{\cosh(\alpha_n b)}\sin(\alpha_n a) \right\}$$

$$Q = \frac{2}{\mu}\frac{dP}{dz} \times \left\{ -\frac{2}{3}a^3 b + \sum_{n=0}^{n=\infty} \frac{4}{a}\frac{(-1)^n}{\alpha_n^5}\tanh(\alpha_n b)(-1)^n \right\}$$

$$Q = \frac{2}{\mu}\frac{dP}{dz} \times \left\{ -\frac{2}{3}a^3 b + \sum_{n=0}^{n=\infty} \frac{4}{a\alpha_n^5}\tanh(\alpha_n b) \right\}$$

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[4] Otherwise note that for example:

$$V_z(x,y,z,t) = \bar{V}_z(x,y) + \tilde{V}_z(x,y,t \pm \frac{z}{c}) = \bar{V}(x,y) + \tilde{V}(x,y)\cos(w(t \pm \frac{z}{c}))$$

$$\frac{\partial^2 V_z}{\partial z^2} = -(\frac{w}{c})^2 \tilde{V}(x,y)\cos(w(t \pm \frac{z}{c})) = -(\frac{2\pi}{L})^2 \tilde{V}_z$$

Where "c" is the speed of pressure wave and "L" is its wavelength. However, note that due to rather incompressible behavior of fluids, the "L" is too large and we can ignore above term (e.g. in water the sound\pressure wave speed is about 1.5 Km/sec).

[5] A very similar problem is solved in following publication: "S. Tsangaris and N. W. Vlachakis", "Journal of Fluids Engineering","Exact Solution of the Navier-Stokes Equations for the Fully Developed, Pulsating Flow in a Rectangular Duct With a Constant Cross-Sectional Velocity ", 2003.

Numerical evaluation of the above summation for square channel (a=b case) leads to:

$$Q \cong \frac{-0.5623}{\mu}\frac{dP}{dz}a^4$$

Note that the 1st term of the summation is equal to $0.3836 \times a^4$ (comparable with out of the summation term $-\frac{2}{3}a^4 = -0.6667 \times a^4$), while the 2nd term is equal to $0.0017 \times a^4$. Consequently the 1st term will suffice very well for calculation of flow rate (due to sharp drop of terms by inverse fifth power).

Note that evaluation of above series with MATLAB is very simple and can be done very quickly:

```
>> a=1;b=1;
>> % Assume dP/dz=1, Mu=1
>> n=[0:200]; % Use 201 terms of series
>> alpha=(n+0.5)*pi/a; % Assume dP/dz=1, Mu=1
>> Q_star=2*(-2/3*a^3*b+4/a*sum(1./alpha.^5.*tanh(alpha*b)))/(-a^3*b)

Q_star =

    0.5623
```

Rather than pressure, the only nonzero components of stresses are:

$$\tau_{yz} = \mu\frac{\partial V_z}{\partial y}$$
$$\tau_{xz} = \mu\frac{\partial V_z}{\partial x}$$

Note that at each point there is a direction attributed to the maximum shear stress. The maximum shear stress is related to the maximum directional gradient of velocity which is equal to the absolute value of the gradient of the velocity:

$$\tau_{MAX}(x,y) = \mu\left|\nabla V_z(x,y)\right|$$
$$\tau_{MAX}(x,y) = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}$$

To compute the shear we need "u" derivatives:

$$\frac{\partial u(x,y)}{\partial x} = 2x - \sum_{n=0}^{n=\infty}\frac{4}{a}\frac{(-1)^n}{\alpha_n^2}\frac{\cosh(\alpha_n y)}{\cosh(\alpha_n b)}\sin(\alpha_n x)$$
$$\frac{\partial u(x,y)}{\partial y} = \sum_{n=0}^{n=\infty}\frac{4}{a}\frac{(-1)^n}{\alpha_n^2}\frac{\sinh(\alpha_n y)}{\cosh(\alpha_n b)}\cos(\alpha_n x)$$

So the shear stress at wall y=b will be:

$$\tau_{yz}(x, y=b) = \frac{1}{2}\frac{dP}{dz}\sum_{n=0}^{n=\infty}\frac{4}{a}\frac{(-1)^n}{\alpha_n^2}\tanh(\alpha_n b)\cos(\alpha_n x)$$

And the shear stress at wall x=a will be:

$$\tau_{xz}(x=a, y) = \frac{1}{2}\frac{dP}{dz}\left\{2a - \sum_{n=0}^{n=\infty}\frac{4}{a}\frac{(-1)^n}{\alpha_n^2}\frac{\cosh(\alpha_n y)}{\cosh(\alpha_n b)}\sin(\alpha_n a)\right\}$$

$$\tau_{xz}(x=a, y) = \frac{1}{2}\frac{dP}{dz}\left\{2a - \sum_{n=0}^{n=\infty}\frac{4}{a\alpha_n^2}\frac{\cosh(\alpha_n y)}{\cosh(\alpha_n b)}\right\}$$

The nature of problem suggest that due to symmetry and due to the fact that a<b, the maximum shear will happen at "x=a, y=0" [6]:

$$\tau_{\max} = \tau_{xz}(x=a, y=0) = \frac{1}{2}\frac{dP}{dz}\left\{2a - \sum_{n=0}^{n=\infty}\frac{4}{a\alpha_n^2}\frac{1}{\cosh(\alpha_n b)}\right\}$$

However, other than physical intuition, it is rather hard to make a clear comparison between shear stress at "x=a" and "y=b" walls (to find the maximum shear stress). So we indeed compare them by plotting their values, shown on the next page. As a result for a=b case the maximum shear will be:

$$\tau_{\max}\Big|_{a=b} \cong 0.675 \times a\frac{dP}{dz}$$

Note that for special case of a=b, the shear profile at both walls should be the same, as well as the velocity profile at x=0 and y=0 planes. Indeed this is a very good point to check our solution and we utilize that in the attached file "C2p29_PSET4_1a.m". From here forwards, we utilize the dimensionless variables for plots and discussions and they are defined accordingly:

$$V_z^* = \frac{V_z}{-\frac{a^2}{\mu}\frac{dP}{dz}}$$

$$\tau^* = \frac{\tau}{a\frac{dP}{dz}}$$

$$Q^* = \frac{Q}{-\frac{a^3 b}{\mu}\frac{dP}{dz}}$$

[6] Note that $\frac{\cosh(\alpha_n y)}{\cosh(\alpha_n b)} \le 1$ and $\sum_{n=0}^{n=\infty}\frac{4}{a\alpha_n^2}$ =2a, so clearly the maximum shear stress at x=a wall happens at y=0.

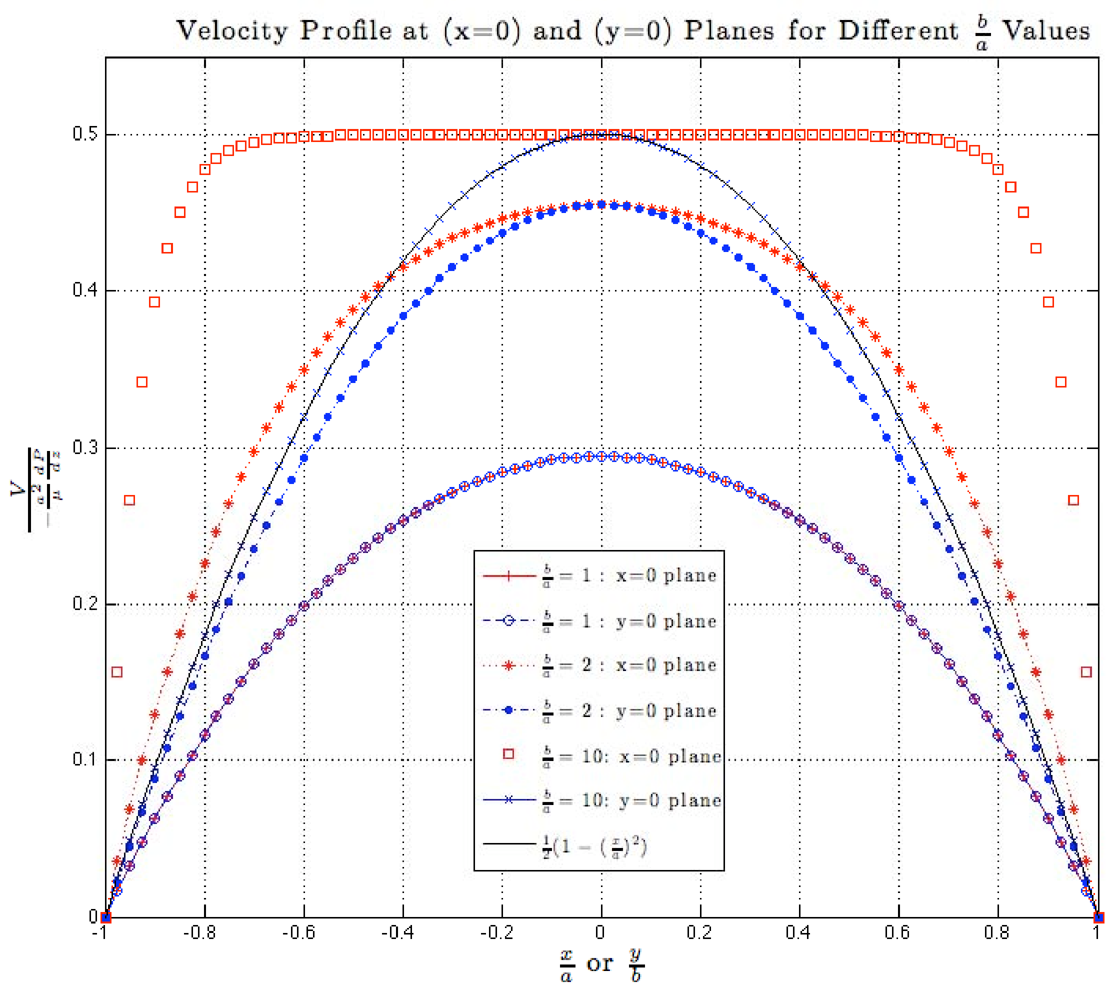
Note that velocity profiles at x=0 and y=0 planes can be computed from below relations:

$$u(x=0,y) = -a^2 + \sum_{n=0}^{n=\infty} \frac{4}{a} \frac{(-1)^n}{\alpha_n^3} \frac{\cosh(\alpha_n y)}{\cosh(\alpha_n b)}$$

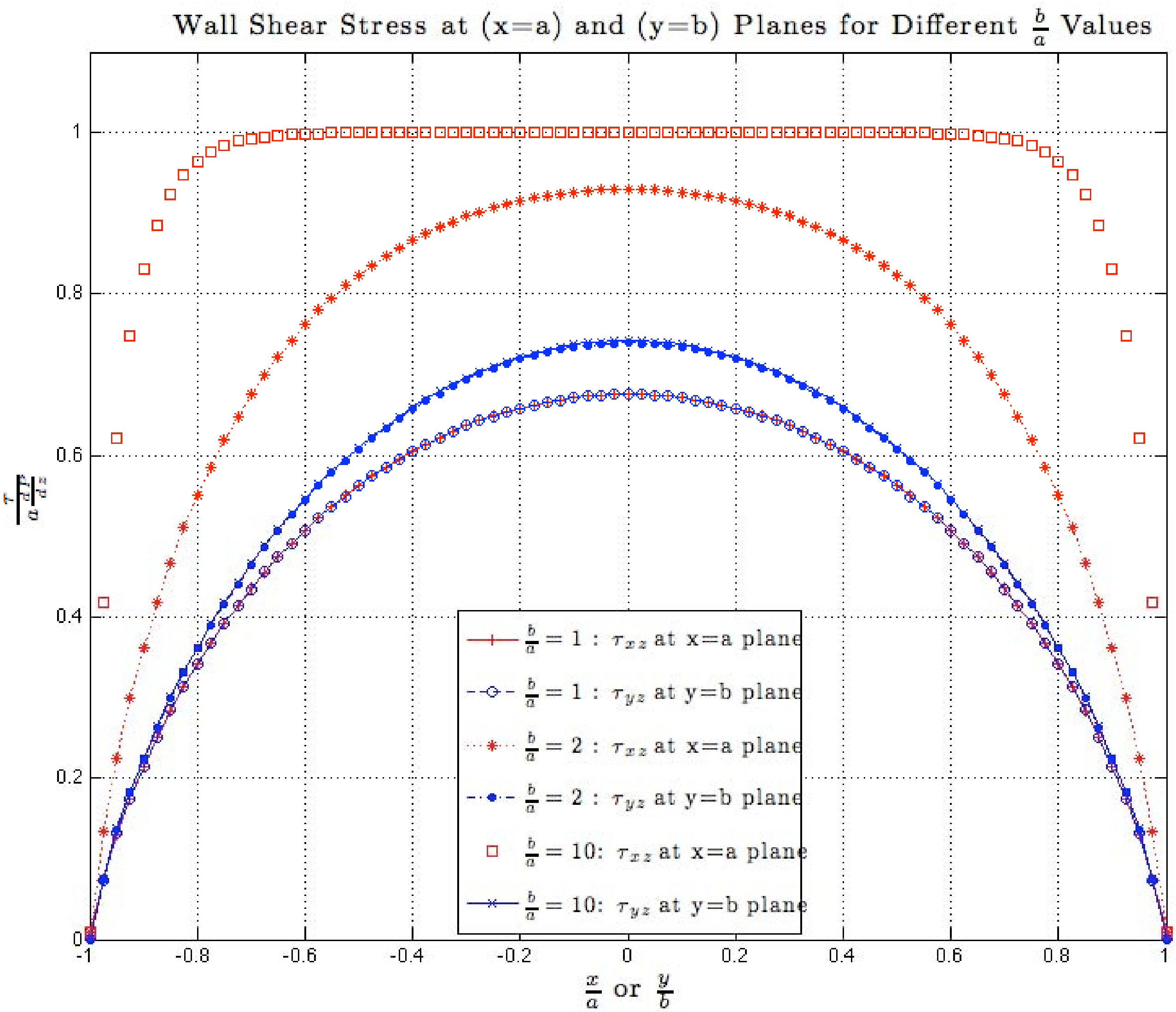
$$u(x,y=0) = (x^2 - a^2) + \sum_{n=0}^{n=\infty} \frac{4}{a} \frac{(-1)^n}{\alpha_n^3} \frac{1}{\cosh(\alpha_n b)} \cos(\alpha_n x)$$

The dimensionless velocity plot is shown here and as we expected for a=b case, both velocity profiles match together (while they both reach a maximum about $\max(V_z^*) \cong 0.3$). However as long as $(\frac{b}{a})$ ratio increases:

- The velocity peak very quickly reaches $\max(V_z^*) \cong 0.5$ (which means that effective friction has decreased).
- The velocity profile at x=0 plane becomes almost flat (like 1D case).
- The velocity profile at y=0 tends to the ideal 1D solution of $V_z^* = \frac{1}{2}(1 - (\frac{x}{a})^2)$



The dimensionless shear stress is plotted below and again as we expected the wall shear stress at both walls match for a=b case. On the other hand, as long as $(\frac{b}{a})$ ratio increases, the τ_{xz} value tends to ideal 1D value equal to $\tau_{xz} = a\frac{dP}{dz}$.



f)

For a circular pipe with similar conditions and a radius equal to "R" the velocity profile will be equal to:

$$V_{z-circular} = \frac{1}{4\mu}\frac{dP}{dz}(r^2 - R^2)$$

$$Q_{circular} = \frac{-\pi R^4}{8\mu}\frac{dP}{dz}$$
$$\tau_{\max-circular} = \frac{R}{2}\frac{dP}{dz}$$

Now note that if the square pipe and the circular pipe have the same area, then we have $4a^2 = 4b^2 = \pi R^2$. Also as problem states they have the same flow rate:

$$Q_{circular} = Q_{square}$$
$$\frac{-0.5623}{\mu}\frac{dP}{dz}a^4(\frac{dP}{dz})_{square} = \frac{-\pi R^4}{8\mu}(\frac{dP}{dz})_{circular}$$
$$\frac{(\frac{dP}{dz})_{square}}{(\frac{dP}{dz})_{circular}} = \frac{\pi R^4}{0.5623\times 8a^4} = \frac{\pi R^4}{0.5623\times 8(\frac{\pi R^2}{4})^2} = \frac{2}{0.5623\times\pi} \cong 1.1322$$

So we can see that the square channel has a rather higher friction for flow compared to circular channel. Also we can compare the maximum shear stress:

$$\frac{\tau_{\max-square}}{\tau_{\max-circular}} = \frac{0.675\times a(\frac{dP}{dz})_{square}}{\frac{R}{2}(\frac{dP}{dz})_{circular}} = 1.1322\times 0.675\times\frac{\sqrt{\pi}\frac{R}{2}}{\frac{R}{2}} \cong 1.354$$

g)

The $o(h^2)$ formulas are utilized for Laplacian approximation and shear stress evaluation:

$$\nabla^2 f_{i,j} \cong \frac{f_{i-1,j} + f_{i+1,j} + f_{i,j-1} + f_{i,j+1} - 4f_{i,j}}{h^2}$$
$$\frac{df_{i,j}}{dx} \cong \frac{3f_{i,j} - 4f_{i-1,j} + f_{i-2,j}}{2h}$$

Symmetry can be incorporated to limit the unknowns to only one quadrant of channel. The attached file “C2p29_PSET4_1b.m” is used to solve the equations. Note that by modulating the program we can simplify our task a lot. As a general advice, it is very good to define two modules to relate double indexes unknowns (i,j) to single indexes (I) and inversely.

The 1st attempt is based only on one central node at x=y=0 and its corresponding flow rate is computed by trapezoidal rule. For finer meshes, the Simpson’s 1/3 rule is utilized for flow rate computation.

The program output is shown here and as can be checked for $n = \frac{a}{h}$ equal to 32 (32^2 unknown nodal velocity), the relative error in flow rate has decreased below desired value equal to 0.1%. At the same time, the shear stress has reached the analytical value.

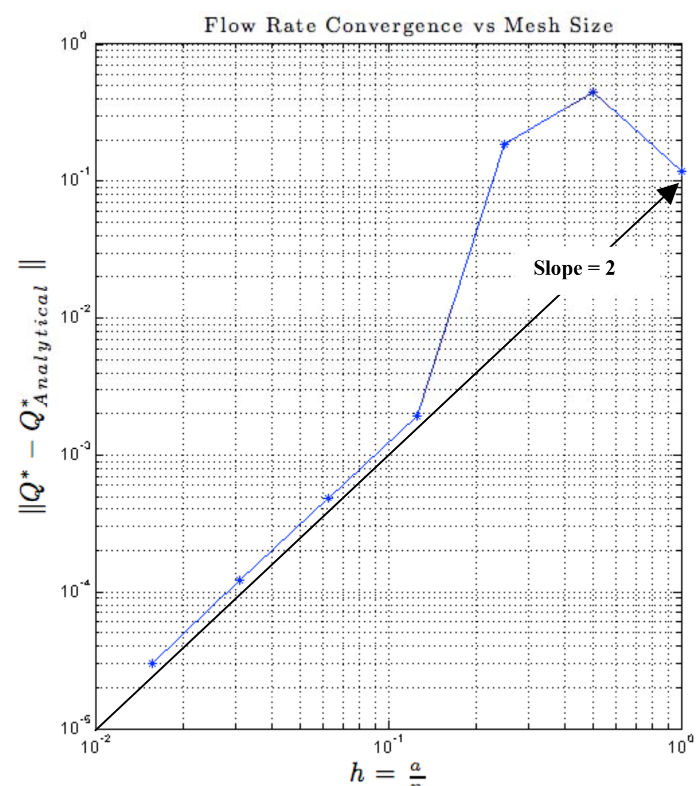
Mesh Density (n=a/h)	Normalized Flow Rate	Flow Rate Relative Error	Maximum Shear Stress
1	+0.44444	NaN%	+0.50000
2	+1.01042	-5.60e+01%	+0.59375
4	+0.74585	+3.55e+01%	+0.65165
8	+0.56037	+3.31e+01%	+0.66927
16	+0.56182	-2.59e-01%	+0.67380
32	+0.56219	-6.45e-02%	+0.67493
64	+0.56228	-1.61e-02%	+0.67522

h)

The numerical value corresponding to the previous step is equal to $Q^*_{num} = 0.56219$. The analytical value is equal to $Q^*_{analytical} = 0.562308$ and as a result the numerical value has a relative error about 0.02%.

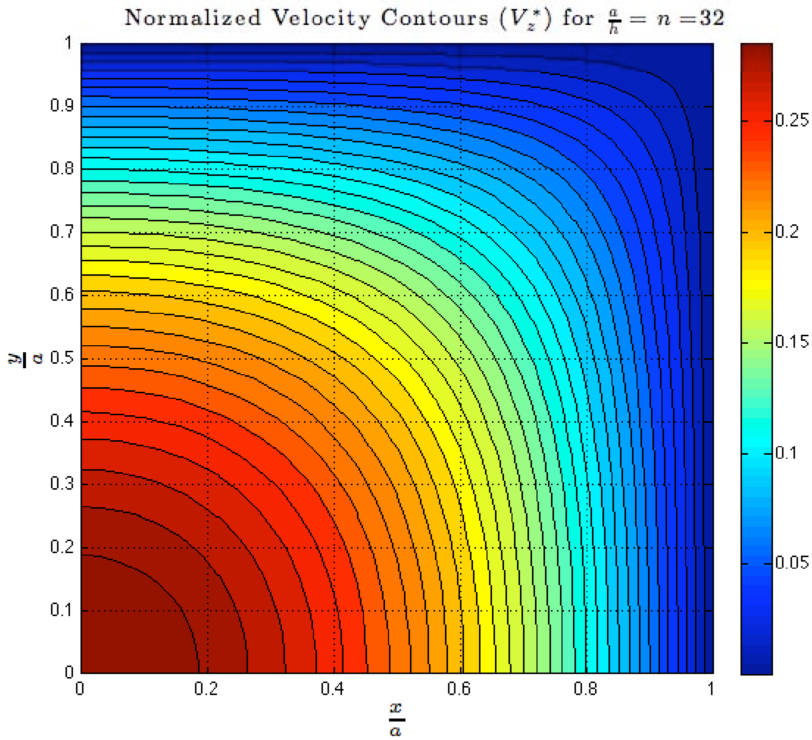
i)

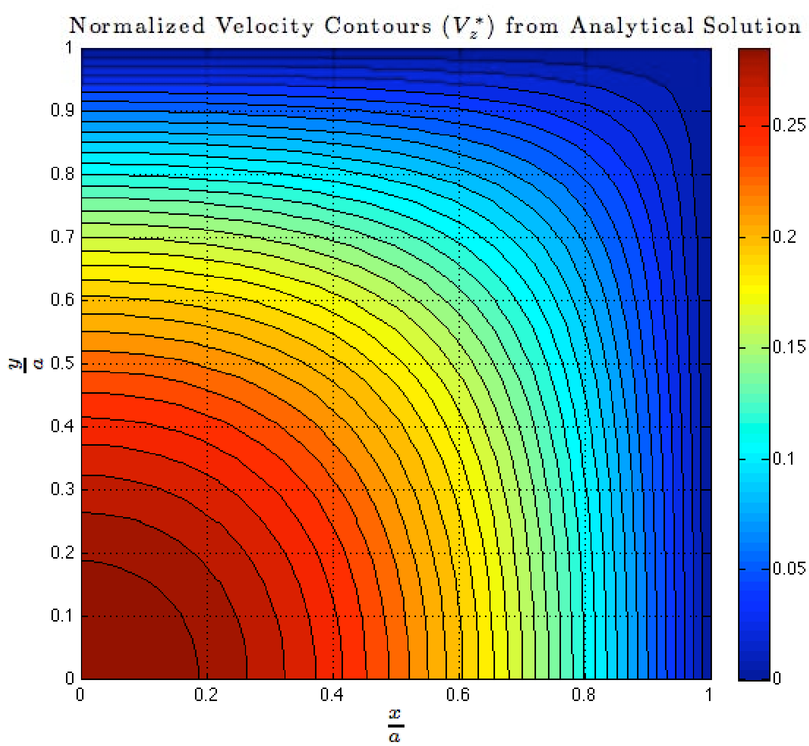
The below graph is computed by the previous file. As it can be seen, the absolute error in the flow rate is proportional to h^2 (the slope can be checked in the logarithmic plot). This indeed corresponds to the $o(h^2)$ accurate solution of velocity field, followed by an $o(h^4)$ accurate integration.



j)

Plots are shown here and as can be seen, both match very well.





k)

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2$$

$$\eta = \frac{x}{a},\ \xi = \frac{y}{b} \Rightarrow dx = a\,d\eta,\ dy = b\,d\xi$$

$$u(x,y) = u(a\eta, b\xi) = U(\eta,\xi)$$

$$\frac{1}{a^2}\frac{\partial^2 U}{\partial \eta^2} + \frac{1}{b^2}\frac{\partial^2 U}{\partial \zeta^2} = 2$$

$$\frac{\partial^2 U}{\partial \eta^2} + \frac{a^2}{b^2}\frac{\partial^2 U}{\partial \zeta^2} = 2a^2$$

l)

The previous file is used again to solve the above equation ($\frac{\partial^2 U}{\partial \eta^2} + \frac{a^2}{b^2}\frac{\partial^2 U}{\partial \zeta^2} = 2a^2$). Note that we generate "n+1" nodes in (η,ξ) domain in each direction, so we will have $\Delta\eta = \Delta\xi = h^* = \frac{1}{n}$. Consequently:

$$\text{Define } \beta = \frac{a^2}{b^2} \Rightarrow \frac{\partial^2 U}{\partial \eta^2} + \frac{a^2}{b^2}\frac{\partial^2 U}{\partial \zeta^2} \cong \frac{U_{i-1,j} + U_{i+1,j} + \beta U_{i,j-1} + \beta U_{i,j+1} - 2(1+\beta)U_{i,j}}{h^{*2}}$$

The solution is shown on the next section. Note that plots correspond to n=32 unknown nodes (as previous part).

m)

The program output is shown here:

```
Q_star_anal =

    1.2493
```

Mesh Density (n=a/h)	Normalized Flow Rate	Flow Rate Relative Error	Maximum Shear Stress
1	+0.88009	NaN%	+0.99010
2	+1.96711	-5.53e+01%	+0.99949
4	+1.57613	+2.48e+01%	+0.99998
8	+1.23428	+2.77e+01%	+1.00000
16	+1.24560	-9.08e-01%	+1.00000
32	+1.24839	-2.24e-01%	+1.00000
64	+1.24908	-5.45e-02%	+1.00000

Accordingly, the numerical value corresponding to n=32 is equal to $Q^*_{num} = 1.2484$. The analytical value is equal to $Q^*_{analytical} = 1.2493$ and as a result the numerical value has a relative error about 0.07%. The error is higher than previous case (which was equal to 0.02%) and this can be due to reduced element density in (x,y) domain (although we have the same element density in (η, ξ) domain).

Note that for numerical evaluation of series, we have to break the "cosh" or "sinh" terms (otherwise individual terms will tend to infinity and blow up):

$$\frac{\cosh(\alpha_n y)}{\cosh(\alpha_n b)} = \frac{e^{\alpha_n y} + e^{-\alpha_n y}}{e^{\alpha_n b} + e^{-\alpha_n b}} = \frac{e^{\alpha_n y}}{e^{\alpha_n b}} \left(\frac{1 + e^{-2\alpha_n y}}{1 + e^{-2\alpha_n b}} \right)$$

$$\frac{\cosh(\alpha_n y)}{\cosh(\alpha_n b)} = e^{\alpha_n (y-b)} \left(\frac{1 + e^{-2\alpha_n y}}{1 + e^{-2\alpha_n b}} \right), \text{ use for } y > 0 \ \text{ (note that } y - b < 0)$$

Also note that:

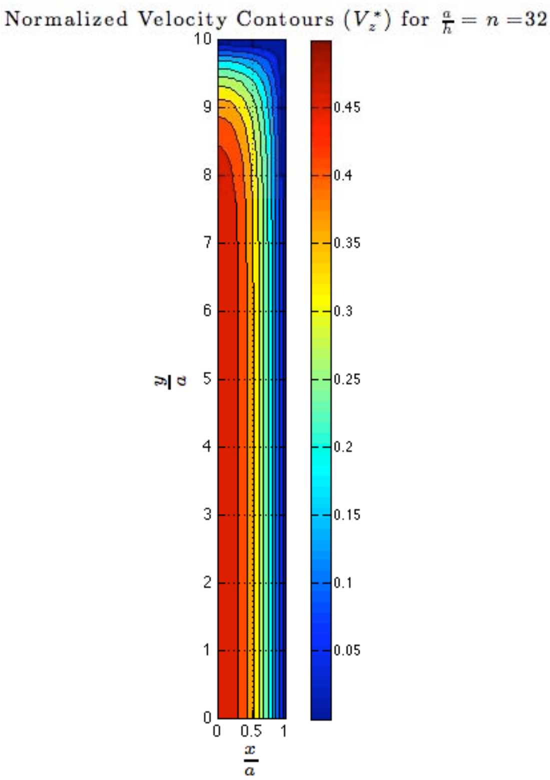
$$e^{\alpha_n (y-b)} = e^{(n+0.5)\pi \frac{(y-b)}{a}}$$

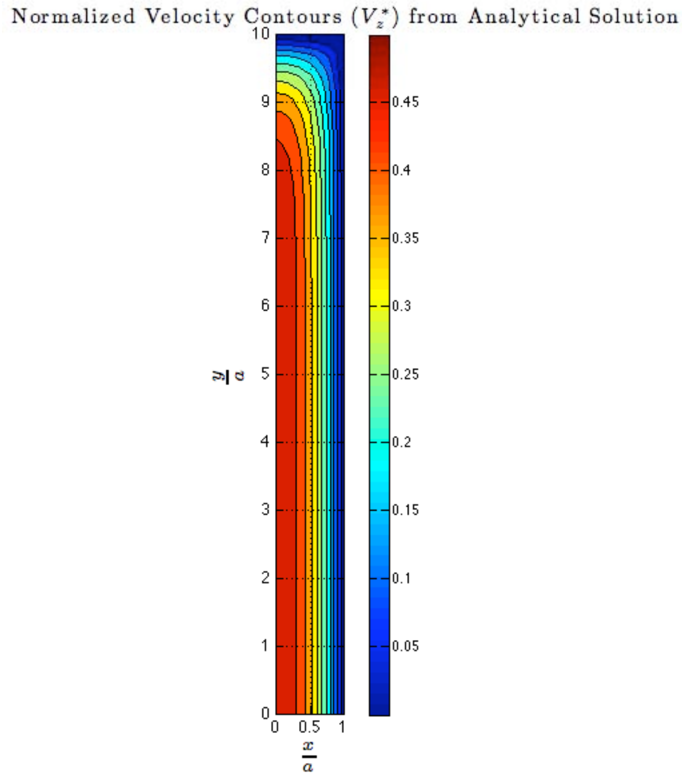
$$\text{Slowest Term due to } n = 0: \ e^{\alpha_0 (y-b)} = e^{\frac{\pi}{2} \frac{(y-b)}{a}}$$

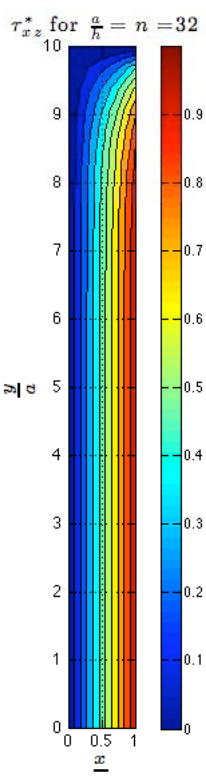
$$e^{\alpha_0 (y-b)} \Big|_{y=b-2a} = e^{-\pi} \cong 0.04$$

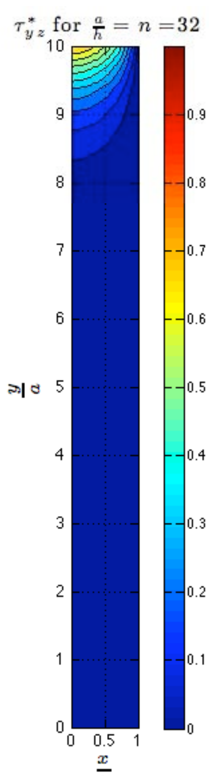
Higher terms decay faster and the solution is governed by 1st term corresponding to n=0. Note that at a distance equal to "2a" from "y" wall (where y=b-2a), the exponential term has decayed from "1.0" to "0.04". This means that at this distance, the "y" edge effect has almost totally decayed (problem is almost 1D). This can be also checked in all previous and next plots for $\frac{b}{a} = 10$.

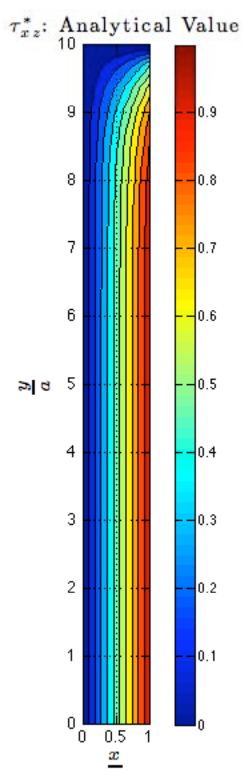
Velocity and shear stress contours are shown on the next page. As can be seen, both analytical and numerical values match very well.

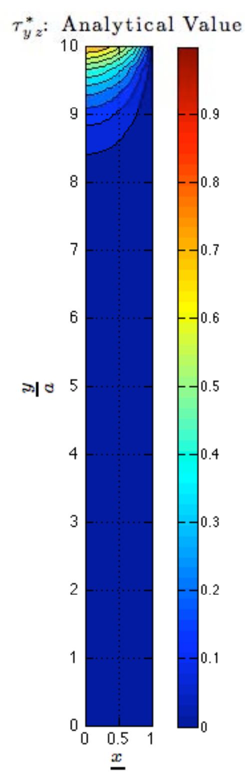

Normalized Velocity Contours (V_z^*) for $\frac{a}{h} = n = 32$
$\frac{y}{a}$
$\frac{x}{a}$


Normalized Velocity Contours (V_z^*) from Analytical Solution
$\frac{y}{a}$
$\frac{x}{a}$


τ_{xz}^* for $\frac{a}{h} = n = 32$
$\frac{y}{a}$
$\frac{x}{a}$


τ_{yz}^* for $\frac{a}{h} = n = 32$
$\frac{y}{a}$
$\frac{x}{a}$

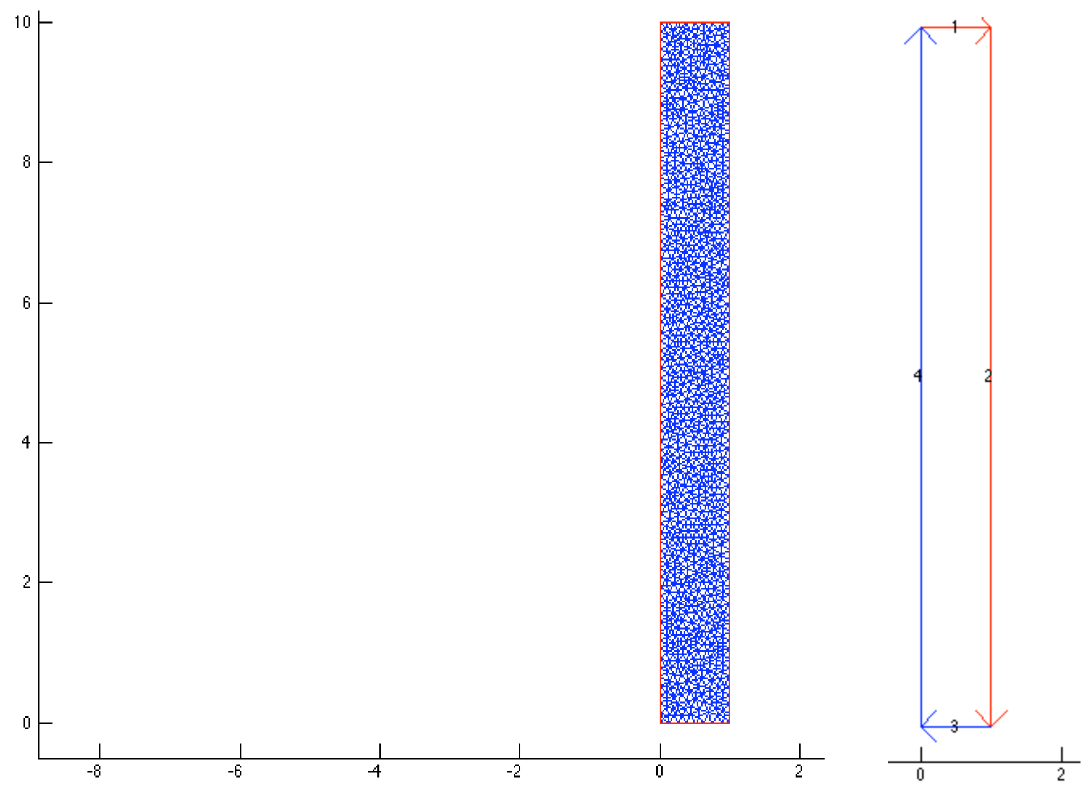

τ_{xz}^*: Analytical Value
$\frac{y}{a}$
$\frac{x}{a}$


τ_{yz}^*: Analytical Value
$\frac{y}{a}$
$\frac{x}{a}$

n)

The MATLAB "pdetool" is used for simulation and the settings are saved in attached "C2p29_PSET4_1c.m" file. A rectangle corresponding to the upper-right quadrant of the channel is drawn and it is meshed by triangular elements. We had $33^2 = 1089$ nodes (but 32^2 unknowns) in the previous step, so the maximum edge size is set to the value of $0.13 \times a$ to achieve a similar number of nodes equal to 1085. While we have used almost the same number of nodes, there could be a noticeable error due to smaller number of nodes in more important direction of "x".

Geometrical Domain and Boundary Edges:

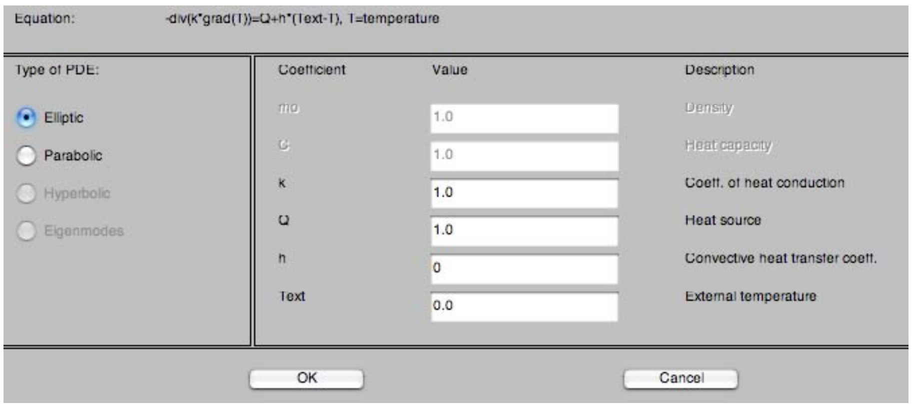


The heat transfer equation is similar to our equation and we use that instead which is already setup in MATLAB pdetool (alternatively we can setup a generic hyperbolic equation). The equation settings are shown in the MATLAB windows on the next page. Note the similarity between thermal domain and our problem:

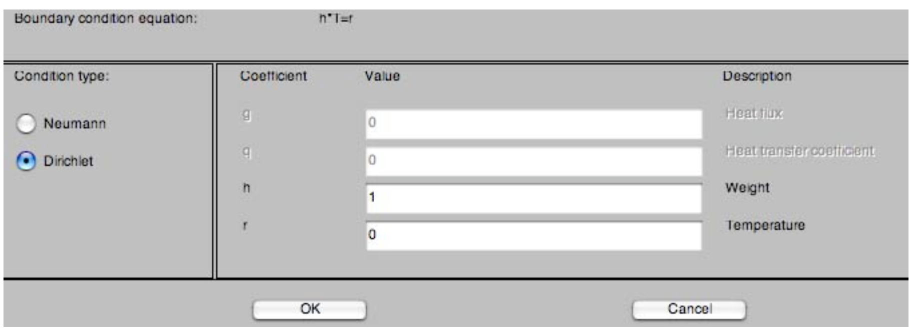
- Temperature (T) $\Rightarrow$ Velocity (V_z)
- Conductivity (k) $\Rightarrow$ Viscosity (μ)
- Heat Source (Q) $\Rightarrow$ Pressure Gradient ($\frac{dP}{dz}$)

To find the normalized velocity solution the "k" and "Q" are set to "1" and "-1". Next we set the boundary conditions. The boundary edges are shown on above right picture. The red edges (number 1 and 2), are set to prescribed zero temperature, which is named as Dirichlet boundary condition. On the other hand, the blue edges (number 3 and 4) are set to zero temperature flux (symmetry implies that $\frac{\partial V_z}{\partial x}$ is zero for edge 4, as well as $\frac{\partial V_z}{\partial y}$ for edge 3). This is indeed named as Neumann boundary condition due to prescribed derivative with respect to the edge unit vector named as $\vec{n}$ (given value of $\frac{\partial T}{\partial \vec{n}} = \nabla T.\vec{n}$).

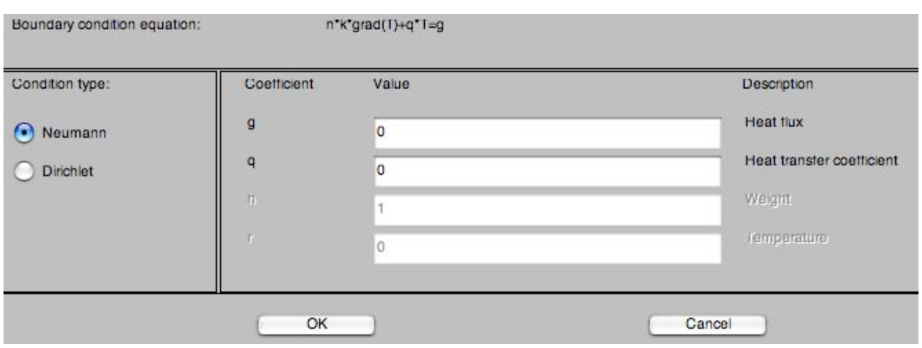
Equation Setting:



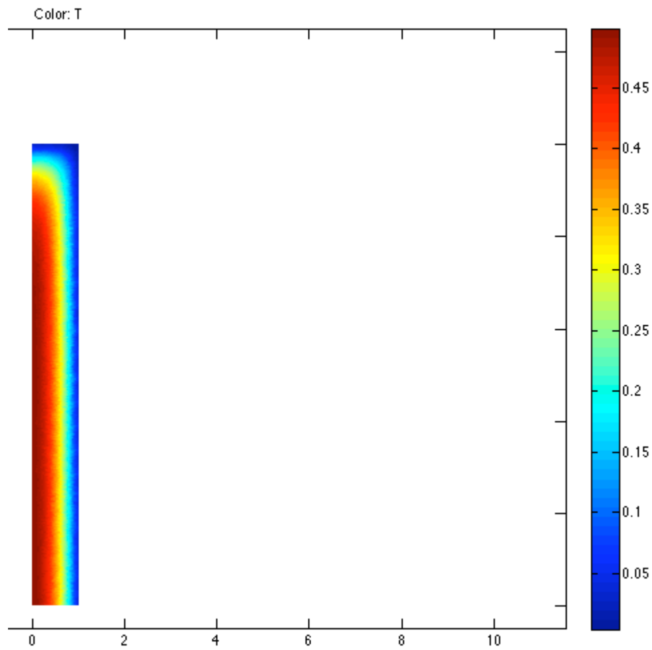
Boundary Conditions for Edge 1-2:

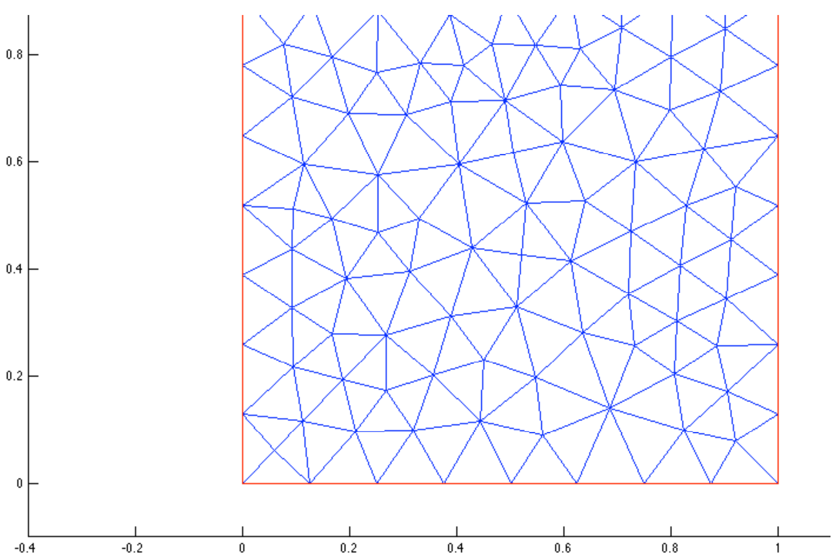


Boundary Conditions for Edge 3-4:



Finally we can solve the equation and attain the normalized velocity distribution as shown below. Note that the general appearance of solutions sounds OK and it is similar to our expected analytical solution. The peak normalized velocity is about 0.50 which is very close to expected value of 0.50 (this was achieved in our previous numerical scheme as well).

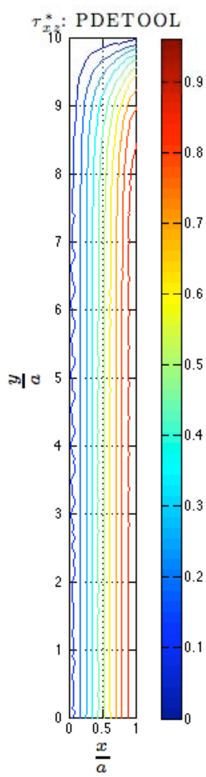


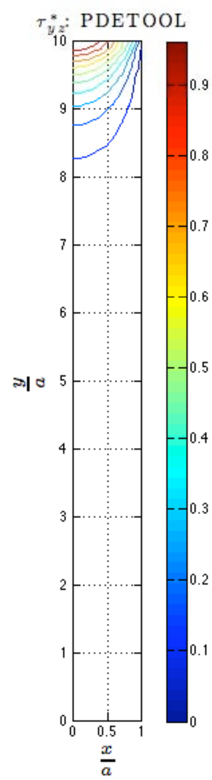


Finally we compute the net flow rate. To do so we export the solution and mesh to the workspace and utilize attached "C2p29_PSET4_1d.m" file. The file simply calculates the area of each triangle, multiply it by average of nodal velocities of triangle vertices and sum it on all triangles. The function "pdetrg" is used to calculate the area of elements from triangulation data and nodal coordinates.

By running above program, we get $Q^*_{pde} = 1.2460$. The analytical value is equal to $Q^*_{analytical} = 1.2493$ and as a result the pdetool numerical value has a relative error about 0.27%.

Shear stress contours are shown below and we have used "pdegrad" to compute the velocity gradients. Again shear stresses have small errors, while analytical value of max(τ^*_{xz}) is equal to 1.0 but here we get 0.96. Also while analytical value of max(τ^*_{yz}) is equal to 0.73 but here we get 0.69 . As a result, we see that the error of MATLAB general pde solver is higher (using the same number of nodes) for our particular problem compared to our written codes. This could be possibly due to lower mesh density in "x" direction. Furthermore, note that the heat flux quantity corresponds to the previously defined maximum shear stress of arbitrary points ($\tau_{MAX}(x,y) = \mu \left|\nabla V_z(x,y)\right|$).





o)

$$Q = \frac{2}{\mu}\frac{dP}{dz} \times \left\{ -\frac{2}{3}a^3 b + \sum_{n=0}^{n=\infty} \frac{4}{a\alpha_n^5} \tanh(\alpha_n b) \right\}$$

$$Q = \frac{2}{\mu}\frac{dP}{dz} \times \left\{ -\frac{2}{3}a^3 b + \frac{4a^4}{\pi^5} \sum_{n=0}^{n=\infty} \frac{1}{(n+0.5)^5} \tanh(\alpha_n b) \right\}$$

$$Q^* = \left\{ \frac{4}{3} - \frac{8}{\pi^5}(\frac{a}{b}) \sum_{n=0}^{n=\infty} \frac{1}{(n+0.5)^5} \tanh(\alpha_n b) \right\}$$

$$\lim_{\frac{a}{b}\to 0} \tanh(\alpha_n b) = 1$$

$$\lim_{\frac{a}{b}\to 0} Q^* = \frac{4}{3}$$

Note that the limiting value of Q^* is not dependent on either "b" or "a". Even for our previous case $Q^* \cong 1.25$, which is rather closes to $\frac{4}{3} \cong 1.33$.

While this approximation, favors 1D simulation it should be noted that edge effects are still present at "y" edges. This indeed presents a severe challenge to capture details around edges. So we have to solve the equation for a very large 2D domain while it is mostly a 1D problem. A remedy can be to solve the problem in a larger $\frac{a}{b}$ ratio. This unrealistic $\frac{a}{b}$ ratio depends on our required accuracy, but usually a ratio about $\frac{1}{5}$ is fine.