2.25 Advanced Fluid Mechanics

Solution to Prob 1, Quiz 1, Fall 2003
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This device is a very clever, simple design that yields information about exhalation pressure and volume flow rate with one single measurement. Blowing into the tube for a few seconds raises the interior pressure *p* and moves the piston outward, which raises the pressure (because of spring extension) and also the volume flow rate also (the open slit area increases), until the piston reaches an equilibrium displacement *x*.

Let the pressure inside the tube be *p*, and the inlet velocity be $V_1 = Q/\pi R^2$. Since the outflow area wx at the slit is very small compared with the inflow area πR^2, the velocity V_1 at the mouth (and thus everywhere inside the tube) is negligible compared with the velocity V_2 at the slit. Applying Bernoulli's equation from the tube to a point in the jet from the slit, just outside the slit, where the pressure is atmospheric (*p*=0), gives

$$V_2 = \sqrt{2p/\rho} \qquad (1)$$

Force balance on the piston, which is essentially static during the measurement of x, implies that

$$p\pi R^2 = kx \qquad (2)$$

or

$$p = \frac{kx}{\pi R^2} \qquad (3)$$

Mass conservation further implies that

$$Q = V_2 wx = \sqrt{\frac{2p}{\rho}} wx \qquad (4)$$

or, inserting (3),

$$Q = \left(\frac{2k}{\rho \pi R^2} \right)^{\frac{1}{2}} wx^{3/2} \qquad (5)$$

Equations (5) and (3) give *Q* and *p* as functions of the piston displacement *x*.

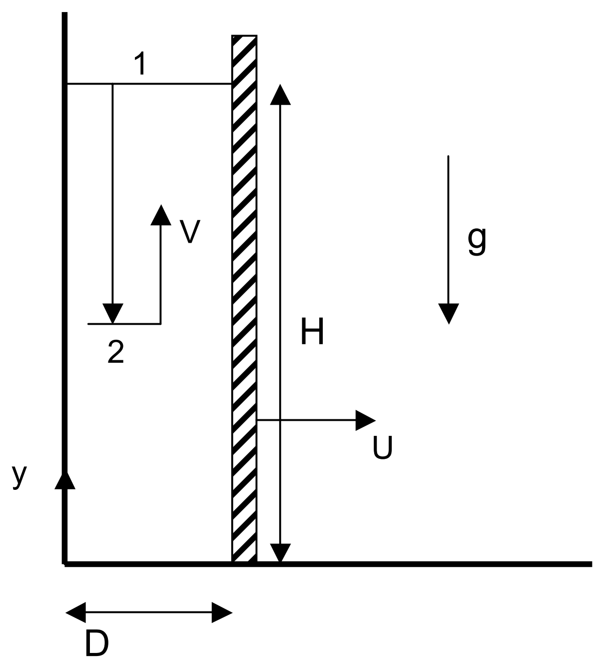
SOME COMMENT

1. Despite numerous questions about whether this Quiz would include the momentum theorem, all answered in the negative both verbally and via e-mail, a large fraction of the class went directly to the momentum theorem—no one successfully. This problem does not lend itself easily to being solved by via the momentum theorem. (Ask at the tutorials.)

2. Whenever you do use the CV approach, indicate *clearly* your CS and (in the case of the momentum theorem) your frame of reference and the forces that are exerted on your CV by the rest of the universe.

2. PLEASE WRITE LEGIBLY, differentiating clearly between p and ρ, etc.

PROBLEM II



The flow is inviscid and incompressible. Since H>>D, we make the approximations that it is one dimensional.

The vertical fluid velocity at any section is $V = V(y,t)$, while the horizontal gate velocity is $U = \frac{\partial D}{\partial t}$.

The total fluid volume is constant, $HWD = Const$, and hence $HD = H_o D_o$.

From mass conservation, $V = -U\frac{y}{D}$. Furthermore, $\frac{\partial V}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{U}{D}\right)y$.

Apply Bernoulli's equation between the free surface and an arbitrary pointy in the fluid, 1 $\rightarrow$2

$$\int_1^2 \frac{\partial \vec{V}}{\partial t} \bullet d\vec{s} + \left[\frac{p}{\rho} + \frac{V^2}{2} + gh\right]_1^2 = 0$$

Note that $\vec{V} \bullet d\vec{s} = V\,dy$, and using the relations above,

$$\frac{\partial (U/D)}{\partial \mathrm{t}} \int_H^y y dy + \frac{p - p_a}{\rho} + \frac{1}{2}\left(\frac{U}{D}\right)^2 \left(y^2 - H^2\right) + g\left(y - H\right) = 0, \text{ after integration:}$$

$$\left(\frac{p - p_a}{\rho}\right) = \frac{1}{2}\left(\frac{U}{D}\right)^2 \left(H^2 - y^2\right) + g\left(H - y\right) + \frac{\partial}{\partial t}\left(\frac{U}{D}\right)\frac{1}{2}\left(H^2 - y^2\right).$$

The force on the gate is the integral of the pressure difference:

$$\frac{F}{W\rho} = \int_o^H \frac{\Delta p dy}{\rho} = \left(\frac{1}{2}\left(\frac{U}{D}\right)^2 + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{U}{D}\right)\right)\left[H^3 - \frac{H^3}{3}\right] + g\left(H^2 - \frac{H^2}{2}\right), \text{ yielding:}$$

$$\frac{F}{PW} = \frac{1}{2}\left(\left(\frac{U}{D}\right)^2 + \frac{\partial}{\partial t}\left(\frac{U}{D}\right)\right) \cdot \frac{2}{3}H^3 + \frac{g}{2}H^2.$$

Now the equation of motion on the gate is $F = Ma$. Substituting:

$$\frac{M}{\rho W}\frac{\partial^2 D}{\partial t^2} = \frac{H^3}{3}\left(\frac{1}{D}\frac{\partial^2 D}{\partial t^2} - \frac{1}{D^2}\left(\frac{\partial D}{\partial t}\right)^2\right) + \frac{g}{2}\frac{H_o^{\,2} D_o^{\,2}}{D^2} = \frac{H_o^{\,3} D_o^{\,3}}{3D^3}\left(\frac{1}{D}\frac{\partial^2 D}{\partial t^2} - \frac{1}{D^2}\left(\frac{\partial D}{\partial t}\right)^2\right) + \frac{g}{2}\frac{H_o^{\,2} D_o^{\,2}}{D^2}$$

and with further reduction we obtain the following differential equation:

$$\left(\frac{H_o^{\,3} D_o^{\,3}}{3}D - \frac{M}{\rho W}D^5\right)\frac{\partial^2 D}{\partial t^2} - \frac{H_o^{\,3} D_o^{\,3}}{3}\left(\frac{\partial D}{\partial t}\right)^2 - \frac{g}{2}H_o^{\,2} D_o^{\,2} D^3 = 0.$$