Problem Set 3

Problem 3.1 (Invariance of coding gain)

(a) Show that in the power-limited regime the nominal coding gain $\gamma_c(\mathcal{A})$ of (5.9), the UBE (5.10) of $P_b(E)$, and the effective coding gain $\gamma_{\rm eff}(\mathcal{A})$ are invariant to scaling, orthogonal transformations and Cartesian products.

(b) Show that in the bandwidth-limited regime the nominal coding gain $\gamma_c(\mathcal{A})$ of (5.14), the UBE (5.15) of $P_s(E)$, and the effective coding gain $\gamma_{\rm eff}(\mathcal{A})$ are invariant to scaling, orthogonal transformations and Cartesian products.

Problem 3.2 (Orthogonal signal sets)

An *orthogonal signal set* is a set $\mathcal{A} = \{\mathbf{a}_j, 1 \le j \le M\}$ of M orthogonal vectors in $\mathbb{R}^M$ with equal energy $E(\mathcal{A})$; *i.e.,* $\langle \mathbf{a}_j, \mathbf{a}_{j'} \rangle = E(\mathcal{A})\delta_{jj'}$ (Kronecker delta).

(a) Compute the nominal spectral efficiency ρ of $\mathcal{A}$ in bits per two dimensions. Compute the average energy E_b per information bit.

(b) Compute the minimum squared distance $d^2_{\min}(\mathcal{A})$. Show that every signal has $K_{\min}(\mathcal{A}) = M - 1$ nearest neighbors.

(c) Let the noise variance be $\sigma^2 = N_0/2$ per dimension. Show that the probability of error of an optimum detector is bounded by the UBE

$$\Pr(E) \le (M-1)Q^{\surd}(E(\mathcal{A})/N_0).$$

(d) Let $M \to \infty$ with E_b held constant. Using an asymptotically accurate upper bound for the $Q^{\surd}(\cdot)$ function (see Appendix), show that $\Pr(E) \to 0$ provided that $E_b/N_0 > 2\ln 2$ (1.42 dB). How close is this to the ultimate Shannon limit on E_b/N_0? What is the nominal spectral efficiency ρ in the limit?

Problem 3.3 (Simplex signal sets)

Let $\mathcal{A}$ be an orthogonal signal set as above.

(a) Denote the mean of $\mathcal{A}$ by $\mathbf{m}(\mathcal{A})$. Show that $\mathbf{m}(\mathcal{A}) \neq \mathbf{0}$, and compute $||\mathbf{m}(\mathcal{A})||^2$.

The zero-mean set $\mathcal{A}' = \mathcal{A} - \mathbf{m}(\mathcal{A})$ (as in Exercise 2) is called a *simplex signal set.* It is universally believed to be the optimum set of M signals in AWGN in the absence of bandwidth constraints, except at ridiculously low SNRs.

(b) For $M = 2, 3, 4$, sketch $\mathcal{A}$ and $\mathcal{A}'$.

(c) Show that all signals in $\mathcal{A}'$ have the same energy $E(\mathcal{A}')$. Compute $E(\mathcal{A}')$. Compute the inner products $\langle \mathbf{a}_j, \mathbf{a}_{j'} \rangle$ for all $\mathbf{a}_j, \mathbf{a}_{j'} \in \mathcal{A}'$.

(d) [Optional]. Show that for ridiculously low SNR, a signal set consisting of $M - 2$ zero signals and two antipodal signals $\{\pm\mathbf{a}\}$ has a lower $\Pr(E)$ than a simplex signal set. [Hint: see M. Steiner, "The strong simplex conjecture is false," IEEE Transactions on Information Theory, pp. 721-731, May 1994.]

Problem 3.4 (Biorthogonal signal sets)

The set $\mathcal{A}'' = \pm\mathcal{A}$ of size $2M$ consisting of the M signals in an orthogonal signal set $\mathcal{A}$ with symbol energy $E(\mathcal{A})$ and their negatives is called a *biorthogonal signal set*.

(a) Show that the mean of $\mathcal{A}''$ is $\mathbf{m}(\mathcal{A}'') = \mathbf{0}$, and that the average energy is $E(\mathcal{A})$.

(b) How much greater is the nominal spectral efficiency ρ of $\mathcal{A}''$ than that of $\mathcal{A}$?

(c) Show that the probability of error of $\mathcal{A}''$ is approximately the same as that of an orthogonal signal set with the same size and average energy, for M large.

(d) Let the number of signals be a power of 2: $2M = 2^k$. Show that the nominal spectral efficiency is $\rho(\mathcal{A}'') = 4k2^{-k}$ b/2D, and that the nominal coding gain is $\gamma_c(\mathcal{A}'') = k/2$. Show that the number of nearest neighbors is $K_{\min}(\mathcal{A}'') = 2^k - 2$.

Problem 3.5 (small nonbinary constellations)

(a) For $M = 4$, the (2×2)-QAM signal set is known to be optimal in $N = 2$ dimensions. Show however that there exists at least one other inequivalent two-dimensional signal set $\mathcal{A}'$ with the same coding gain. Which signal set has the lower "error coefficient" $K_{\min}(\mathcal{A})$?

(b) Show that the coding gain of (a) can be improved in $N = 3$ dimensions. [Hint: consider the signal set $\mathcal{A}'' = \{(1,1,1),(1,-1,-1),(-1,1,-1),(-1,-1,1)\}$.] Sketch $\mathcal{A}''$. What is the geometric name of the polytope whose vertex set is $\mathcal{A}''$?

(c) Give an accurate plot of the UBE of the $\Pr(E)$ for the signal set $\mathcal{A}''$ of (b). How much is the effective coding gain, by our rule of thumb and by this plot?

(d) For $M = 8$ and $N = 2$, propose at least two good signal sets, and determine which one is better. [Open research problem: Find the optimal such signal set, and prove that it is optimal.]

Problem 3.6 (Even-weight codes have better coding gain)

Let $\mathcal{C}$ be an (n,k,d) binary linear code with d odd. Show that if we append an overall parity check $p = \sum_i x_i$ to each codeword $\mathbf{x}$, then we obtain an $(n+1,k,d+1)$ binary linear code $\mathcal{C}'$ with d even. Show that the nominal coding gain $\gamma_c(\mathcal{C}')$ is always greater than $\gamma_c(\mathcal{C})$ if $k > 1$. Conclude that we can focus primarily on linear codes with d even.