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From Propositions To Fuzzy Logic and Rules

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Motivation

Present a formal foundation for

- propositional rules
- fuzzy sets
- fuzzy rules

in order to enable understanding and implementation of a fuzzy propositional rules classifier.

Outline

- Propositions
- Propositions over sets
- Fuzzy Sets
- Propsitions over Fuzzy Sets

Propositional Logic

What is a proposition anyway?

A proposition is a statement that is either true or false. In this context, an interesting statement was made by the greek philosopher Eubulides a long time ago:

This statement is false.

Before we can start saying anything about the above or other statements, we need to establish a language, the *propositional language* or *PL*.

Propositional Logic Syntax

Components

The *PL* language consists of

- an infinite set of *variables* $V = \{a, b, \ldots\}$, and
- a set of *symbols* $S = \{\sim, \vee, (,)\}$.

Definition

An *expression* in *PL* is any string consisting of elements from the sets *V* and *S*, i.e., any string of variables and symbols.

Formation Rules

An expression is either a *well formed formula* (wff) or it is not. The following wff fomation rules allow us to define wff:

Definition

- A variable alone is a wff
- If α is a wff, so is $\sim \alpha$, and
- If α and β are wff, so is $(\alpha \vee \beta)$

Example

for variables *a* and *b* the expression $(a \vee \sim\sim\sim b)$ is a wff, while the expression $a \sim \vee b$ is not.

Leaving out outer parentheses

We sometimes leave out the outermost parentheses of expressions: $(\alpha \vee \beta)$ becomes $\alpha \vee \beta$ even though this is, strictly speaking, not a well formed formula according to the rules above.

Propositional Logic Semantics

Semantics = Meaning

Given a wff we would like to determine whether this expression is true or false. In order to do this we need to define the *semantics* or meaning of our language.

Propositional Logic Semantics
Setting: variable value assigments

Definition
We define a *setting* s as a function $s: V \to \{0,1\}$ assigning to each variable either the value 0 or the value 1, denoting true or false respectively.

Propositional Logic Semantics
Interpretation: Truth Value of Expressions

Definition
An *interpretation* is a function that takes as input a wff and returns 0 or 1 depending on the setting used.

- Formally if we let *WFF* denote the (infinite) set of wff of *PL* we define the interpretation I_s as $I_s: WFF \to \{0,1\}$.
- If the setting s is given by the context or is irrelevant, we drop the subscript and just write I.

Propositional Logic Semantics
Semantics of Operators: Pronunciation

$\sim$ and $\vee$ are *propositional operators* and are called *negation* and *disjunction*, respectively.
The expression $\alpha \vee \beta$ is called the "disjunction of α and β", while $\sim \alpha$ is called the "negation of α".
In everyday language negation is often pronounced "not", while disjunction is pronounced "or".

Propositional Logic Semantics
Semantics of Operators: Formals

$$\sim: \{0,1\} \to \{0,1\}$$
$$\vee: \{0,1\} \times \{0,1\} \to \{0,1\}$$

$$\sim(0) = 1$$
$$\sim(1) = 0$$

$\vee$	0	1
0	0	1
1	1	1

Table: Truth table for disjunction $\vee$

Propositional Logic Semantics

Semantics of Operators: Infix notation

Usually the propositional operators taking two arguments (binary operators) are written in what is called infix notation, i.e., instead of $\vee(0,1)$ we write $0 \vee 1$. We also usually remove the parentheses from $\sim(0)$ and write ~ 0.

Example

$\vee(0,1) = 0 \vee 1 = 1$.

Propositional Logic Semantics

Computing the Interpretation I

The computation of I applied to a wff is made according to these rules:

- For a variable a, $I(a) = s(a)$,
- $I(\sim \alpha) = \sim I(\alpha)$, and
- $I(\alpha \vee \beta) = I(\alpha) \vee I(\beta)$

for wff α and β.

Propositional Logic Semantics

Example: Computing the Interpretation I

Example

$$\begin{aligned} I(\sim(\sim a \vee \sim b)) &= \sim I(\sim a \vee \sim b) \\ &= \sim(\sim I(a) \vee \sim I(b)) \\ &= \sim(\sim s(a) \vee \sim s(b)) \end{aligned}$$

If we let $s(a) = 1$ and $s(b) = 0$, then

$$\begin{aligned} I_s(\sim(\sim a \vee \sim b)) &= \sim(\sim 1 \vee \sim 0) \\ &= \sim(0 \vee 1) = \sim 1 = 0 \end{aligned}$$

Propositional Logic

Syntactic "Sugar"

- $\wedge$ is called *conjunction* ("and")

$$(a \wedge b) \stackrel{\text{def}}{=} \sim(\sim a \vee \sim b)$$

  $(a \wedge b)$ is often called the "conjunction of a and b".
- $\rightarrow$ is called *implication* ("implies")

$$(a \rightarrow b) \stackrel{\text{def}}{=} (\sim a \vee b)$$

  Left side is the *antecedent*, right side is the *consequent*. We also let $(b \leftarrow a) \stackrel{\text{def}}{=} (a \rightarrow b)$.
- $\leftrightarrow$ is called *equivalence* ("equivalence")

$$(a \leftrightarrow b) \stackrel{\text{def}}{=} (a \rightarrow b) \wedge (b \rightarrow a)$$

Propositional Logic

Validity and Satisfiability: Defined

A wff α is *valid* if and only if $I_s(\alpha) = 1$ for every setting *s*. A wff α is *satisifiable* if there exists a setting *s* such that $I_s(\alpha) = 1$, and *unsatisfiable* if no such setting *s* exists.

Example

The wff $(\alpha \vee \sim \alpha)$ is valid, while $(\alpha \wedge \sim \alpha)$ is unsatisfiable.

Propositional Logic

Testing for validity: Truth Table Method

The truth table for $(a \rightarrow b)$ is given here:

a	b	$(a \rightarrow b)$
0	0	1
0	1	1
1	0	0
1	1	1

- Table rows represent settings of variables *a* and *b* and the resulting value for $(a \rightarrow b)$.
- Is $(a \rightarrow b)$ valid? Satisfiable?
  Valid: No. Satisfiable: Yes.

Note:Tables can become Large.

Testing for validity: Falsifying Setting Method

Based on the observation that:

- $\sim \alpha$ is satisifiable $\Rightarrow \alpha$ is not valid, or
- $\sim \alpha$ is unsatisifiable $\Rightarrow \alpha$ is valid.

Strategy: find consistent satisfying setting *s* for $\sim \alpha$ or show that there is none.

Propositional Logic

Example: Testing for validity using Falsifying Setting Method

Is $((p \wedge (p \leftrightarrow (q \wedge r))) \rightarrow q)$ valid?

((	p	$\wedge$	(	p	$\leftrightarrow$	(	q	$\wedge$	r	)))	$\rightarrow$	q	)
	1	1		1	1		1	1	1		0	0	
	4	2		6	5		8	7	9		1	3	

Propositional Logic

Example: Testing for validity using Falsifying Setting Method

Is $((p \wedge (p \leftrightarrow (q \wedge r))) \rightarrow q)$ valid?

((	p	$\wedge$	(	p	$\leftrightarrow$	(	q	$\wedge$	r	)))	$\rightarrow$	q	)
	1	1		1	1		$\underline{1}$	1	1		0	$\underline{0}$	
	4	2		6	5		8	7	9		1	3	

Answer: Yes. The settings underlined pose a contradiction.

Note:
If we during the process shown are allowed alternatives, we need to show a contradiction in *all* the possible alternative settings in order to declare our expression valid.

Propositional Logic

The *PL* Logic System: Components

The logic system of *PL* consists of three things:

- The specification of the language *PL*, as given above,
- the set of valid wff of *PL*, known as *axioms*, and
- the two transformation rules *Uniform Substitution* (US) and *Modus Ponens* (MP).

The axioms and wff obtained from the axioms by application of the transformation rules are called *theorems* of *PL*. We denote that α is a theorem by writing $\vdash \alpha$.

Propositional Logic

The *PL* Logic System: Uniform Substitution

- The result of uniformly replacing any variable $a_1, a_2, \ldots, a_n$ in a theorem α with any wff $\beta_1, \beta_2, \ldots, \beta_n$ respectively is itself a theorem.
- Uniform means here that any occurrence of a_i in α is substituted with the same wff β_i. We write this as $\alpha[\beta_1/a_1, \beta_2/a_2, \ldots, \beta_n/a_n]$.

Example
The result of $(a \rightarrow (a \vee b))[(c \wedge d)/a, c/b]$ is $((c \wedge d) \rightarrow ((c \wedge d) \vee c))$.

Propositional Logic

The *PL* Logic System: Modus Ponens

Modus Ponens (also called the rule of detachment) is sometimes written as

$$\frac{\begin{array}{c}\alpha \\ \alpha \rightarrow \beta\end{array}}{\beta}$$

If α and $\alpha \rightarrow \beta$ are theorems, then by MP so is β. This simply reflects the truth-functional meaning of $\rightarrow$.

Propositional Logic

The *PL* Logic System: Derivability

We express the derivability of a wff by one or more wff by '$\Rightarrow$'. As in:

US: $\vdash \alpha \Rightarrow \vdash \alpha[\beta_1/a_1, \beta_2/a_2, \ldots, \beta_n/a_n]$

MP: $\vdash \alpha, (\alpha \rightarrow \beta) \Rightarrow \vdash \beta$

Propositional Logic

Propositional Consequence: Definition

Clear:

we can manipulate wff by using the rules defining operators and semantics.

Definition

The wff β is a *propositional consequence* of wff α if and only if $\alpha \leftrightarrow \beta \wedge \gamma$ for some wff γ.

We formulate this as a derived transformation rule:

PC: $\vdash \alpha, \vdash (\alpha \leftrightarrow (\beta \wedge \gamma)) \Rightarrow \vdash \beta$

Propositional Logic

Propositional Consequence: Proof of Rule

By showing how we would do without the rule:

(1) α^{given}

(2) $(\alpha \leftrightarrow (\beta \wedge \gamma))^{\text{given}}$

(3) $((\alpha \rightarrow (\beta \wedge \gamma)) \wedge ((\beta \wedge \gamma) \rightarrow \alpha))^{\text{US } (a \rightarrow b) \wedge (b \rightarrow a)}$

(4) $((((\alpha \rightarrow (\beta \wedge \gamma)) \wedge ((\beta \wedge \gamma) \rightarrow \alpha))) \rightarrow ((\alpha \rightarrow (\beta \wedge \gamma)))^{\text{US } ((a \wedge b) \rightarrow a)}$

(5) $(\alpha \rightarrow (\beta \wedge \gamma))^{\text{MP } (3)+(4)}$

(6) $(\beta \wedge \gamma)^{\text{MP } (1)+(5)}$

(7) $((\beta \wedge \gamma) \rightarrow \beta)^{\text{US } ((a \wedge b) \rightarrow a)}$

(8) β

Propositional Logic

Propositional Consequence: Example

If Alf studies, Alf gets good grades. If Alf does not study, Alf has a good time. If Alf does not get good grades, Alf does not have a good time.

What can we say about Alf?

Propositional Logic

Propositional Consequence: Example Formals

- s = “Alf studies”
- g = “Alf gets good grades”
- t = “Alf has a good time”

$$(s \rightarrow g) \wedge (\sim s \rightarrow t) \wedge (\sim g \rightarrow \sim t) \leftrightarrow g \wedge (s \vee t)$$

Using PC, we can conclude that Alf gets good grades.

Propositions over Sets

Characteristic Function: Defined

Definition

A *characteristic function* is a function that has as co-domain the set $\{0, 1\}$, i.e., $f: U \rightarrow \{0, 1\}$ is a characteristic function.

- Furthermore, f is the characteristic function of the subset S of U such that S consists exactly of the elements x in U such that $f(x) = 1$.
- Formally, $S = f^{-1}(1) = \{x \in U | f(x) = 1\}$. We will denote the characteristic function for the set $S \subseteq U$ as χ_S.

Propositions over Sets

Propositions: Defined

Now, a proposition over a set is a proposition that describes a property of the elements of that set. Such propositions are modeled by characteristic functions.

Example

Let $\mathbb{N}$ be the set of natural numbers, and let p be the proposition “x is an even number”. We model p by the characteristic function $even: \mathbb{N} \rightarrow \{0, 1\}$ defined as

$$even(x) = (x + 1) \bmod 2$$

We have that $even(2) = 1$, and $even(3) = 0$, and so forth...

Propositions over Sets

Syntax

As before we have to define the language $PL(U)$ of propositions over the set U. Syntactically, this language is identical to the language PL, except that the set V is the set F consisting of (the names of) characteristic functions on the set U.

Propositions over Sets

Semantics: Truth Sets

The semantics of p over U is based on truth sets. We define truth sets of wff of $PL(U)$ according to the following rules: For $p \in F$, and wff α and β

- $T(p) = \{x \in U | p(x) = 1\}$,
- $T(\sim \alpha) = U - T(\alpha)$, and
- $T(\alpha \vee \beta) = T(\alpha) \cup T(\beta)$.

Propositions over Sets

Semantics: Truth Sets for "Syntactic Sugar"

Analogous to the *PL* case:

- $T(\alpha \wedge \beta) = T(\alpha) \cap T(\beta)$,
- $T(\alpha \rightarrow \beta) = (U - T(\alpha)) \cup T(\beta)$, and
- $T(\alpha \leftrightarrow \beta) = ((U - T(\alpha)) \cup T(\beta)) \cap ((U - T(\beta)) \cup T(\alpha))$.

Example

For the natural numbers and the proposition *even* the truth set is $T(even) = \{2, 4, 6, \ldots\}$.

Propositions over Sets

Semantics: Interpretation

If we let $WFF(U)$ be the set of wff of $PL(U)$ we define the interpretation $I(\alpha, x)$ of a wff α with respect to an element x in U to be

- $I(\alpha, x) = 1$ if and only if $x \in T(\alpha)$.

Alternatively, we can formulate the above as

$$I(\alpha, x) = \chi_{T(\alpha)}(x).$$

Propositions over Sets

Interpretation Example

Consider the propositions "x is a prime number" and "x is an even number" over the natural numbers modeled by the characteristic functions *even* and *prime* with the usual definitions. Let $\alpha = even \wedge prime$. Then we have that

$$T(\alpha) = T(even) \cap T(prime) = \{2\},$$

and $I(\alpha, x) = 1$ if and only if $x = 2$.

Propositions over Sets

$PL(U) \supseteq PL$

We state that PL is "contained in" $PL(U)$. Indeed, PL is contained in $PL(\{0, 1\})$ as we can let $a \in V$ become $a \in F$ given by $T(a) = \{s(a)\}$. Then $I_s(\alpha) = I(\alpha, 1)$.

Propositional Rules

The implication view

if-then form

if height = tall and hair = dark then look=handsome

- "height = tall and hair = dark" is the *antecedent* or "if-part",
- "look=handsome" is *consequent*, or "then-part".

Application

- fact: height = tall and hair = dark
- rule: if height = tall and hair = dark then look=handsome
- inference: look=handsome

In effect we are using Modus ponens

Propositional Rules

The implication view: formals

The *descriptor* "height = tall" is a proposition HeightTall over the set of all people. We now formulate the if-then rule as propositions over sets:

$$(\text{HeightTall} \wedge \text{HairDark}) \rightarrow \text{LookHandsome}$$

The application becomes:

$$\frac{\begin{array}{l}(\text{HeightTall} \wedge \text{HairDark}) \\ (\text{HeightTall} \wedge \text{HairDark}) \rightarrow \text{LookHandsome}\end{array}}{(\text{LookHandsome})}$$

Effect:

We infer the unknown proposition LookHandsome.

Propositional Rules

Computation: Computing the Interpretation

Definition

Given a rule $(\alpha \rightarrow \beta)$. The application of this rule to a data point x is the computation of $I(\beta, x)$ as $I(\alpha, x)$.

In other words we set $I(\beta, x) = \begin{cases} 1 & I(\alpha, x) = 1, \\ 0 & \text{otherwise.} \end{cases}$

Fuzzy Sets
Inherent Vagueness

- What would you answer if I ask “Am I tall?”.
- Does knowing that I am 6ft tall help?
- Not really. The problem lies in the meaning of the word “tall”. I might be tall in Japan, but not in Holland.

Inherent Vagueness
Fuzzy sets offer a way of modeling *Inherent Vagueness*.

Fuzzy Sets
Generalization of Characteristic Functions

Central:
the generalization of the characteristic function $\chi_S : U \to \{0,1\}$ of set S to *membership function* $\mu_S : U \to [0,1]$.

$\mu_S : U \to [0,1]$ gives a *degree* of memberhip in the *fuzzy* set S.

Fuzzy Sets
Crisp Set Operators Definitions

Let A and B be two subsets of some set U. We define union, intersection, difference, and complementation using in terms of χ_A and χ_B as follows:

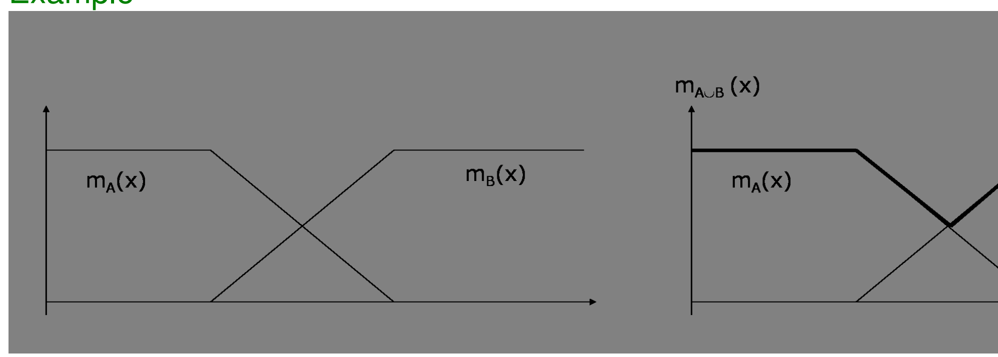
Definition

$$
\begin{aligned}
\chi_{A\cap B}(x) &= \min(\chi_A(x), \chi_B(x)) \\
\chi_{A\cup B}(x) &= \max(\chi_A(x), \chi_B(x)) \\
\chi_{A-B}(x) &= \min(\chi_A(x), 1-\chi_B(x)) \\
\chi_{A^c}(x) &= 1-\chi_A(x).
\end{aligned}
$$

For fuzzy set operations substitute μ for χ.

Fuzzy Sets
Fuzzy Set Operations Example

Example



Fuzzy Relations
Definition

Definition

A fuzzy relation R from a set X to a set Y is a fuzzy set in the cartesian product $X \times Y$, i.e., μ_R is a function $\mu_R : X \times Y \to [0,1]$.

For $x \in X$ and $y \in Y$, the value $\mu_R(x,y)$ gives the degree to which x is related to y in R.

Fuzzy Relations
Crisp Composition

For crisp binary relations $R \subseteq X \times Y$ and $R' \subseteq Y \times Z$ we can formulate their composition in terms of characteristic functions

$$\chi_{R \circ R''}(x,z) = \max_{y \in Y}\{\min(\chi_R(x,y), \chi_{R'}(y,z))\}$$

Fuzzy Relations
Fuzzy Composition

Definition

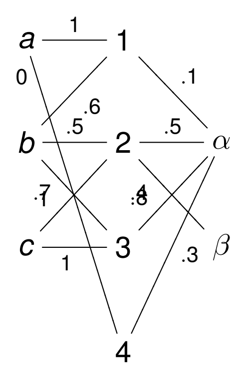
Let X, Y and Z be three sets and let R and R' be two fuzzy relations from X to Y and Y to Z, respectively.

$$\mu_{R \circ R''}(x,z) = \max_{y \in Y}\{\min(\mu_R(x,y), \mu_{R'}(y,z))\}.$$

For fuzzy relations, the definition of composition is essentially identical to the crisp case but for the now expected substitution of μ for χ.

Example

Consider sets $X = \{a,b,c\}$, $Y = \{1,2,3,4\}$, and $Z = \{\alpha, \beta\}$ and fuzzy relations R and R' from X to Y and Y to Z respectively. Diagram:



$$\begin{aligned}\mu_{R \circ R'}(b,\alpha) &= \max\{\min(.6,.1)), \\ &\quad \min(.5,.5), \\ &\quad \min(.7,.8)\} \\ &= \max\{.1,.5,.7\} = .7\end{aligned}$$

Fuzzy Logic

Defining the Fuzzy Logic Language

Recall:

For $PL(U)$, the interpretation $I(\alpha, x)$ is given by

$$I(\alpha, x) = \chi_{T(\alpha)}(x).$$

Definition (Fuzzy Propositional Language)

We define $FPL(U)$, the language of propositions over fuzzy sets by substituting μ for χ in the definition of $PL(U)$.

Fuzzy Logic

Semantics

Definition (Fuzzy Truth Set)

We define the fuzzy truth set $T(\alpha)$ of wff α in $PL(U)$ according to the following rules. For $p \in F$, $x \in U$, and wffs α and β:

- $\mu_{T(p)}(x) = p(x)$,
- $\mu_{T(\sim\alpha)}(x) = 1 - \mu_{T(\alpha)}(x)$, and
- $\mu_{T(\alpha \vee \beta)}(x) = \max(\mu_{T(\alpha)}(x), \mu_{T(\beta)}(x))$.

Fuzzy Logic

Semantics of "Syntactic Sugar"

Analogous to the $PL(U)$ case we can show that

- $\mu_{T(\alpha \wedge \beta)}(x) = \min(\mu_{T(\alpha)}(x), \mu_{T(\beta)}(x))$,
- $\mu_{T(\alpha \rightarrow \beta)} = \max(1 - \mu_{T(\alpha)}(x), \mu_{T(\beta)}(x))$, and
- $\mu_{T(\alpha \leftrightarrow \beta)}(x) =$ $\min(\max(1 - \mu_{T(\alpha)}(x), \mu_{T(\beta)}(x)), \max(1 - \mu_{T(\beta)}(x), \mu_{T(\alpha)}(x)))$.

Definition (Fuzzy Interpretation)

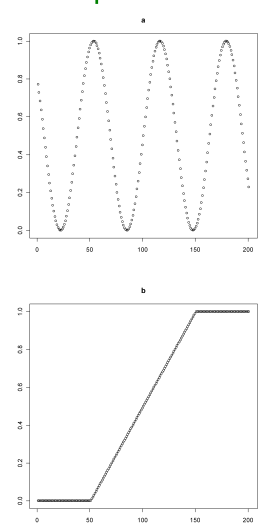
If we let $WFF(U)$ be the set of wffs of $FPL(U)$ we define the interpretation $I(\alpha, x)$ of a wff α with respect to an element x in U to be

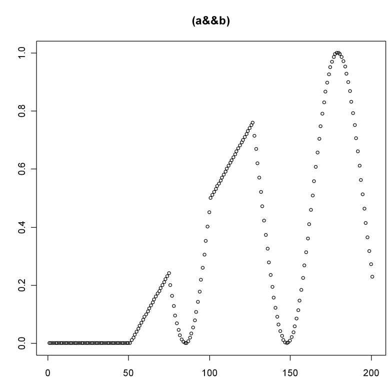
- $I(\alpha, x) = \mu_{T(\alpha)}(x)$.

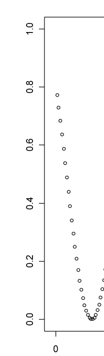
Fuzzy Logic

Examples

Example







Fuzzy Rules
Definition

There are different ways of defining fuzzy rules. We choose the following:

Definition
$I(\beta, x)$ is computed as

$$I(\beta, x) = I(\alpha, x)$$

according to the fuzzy rule $(\alpha \to \beta)$.

Summary
Propositions

We have learned

- about the propositional language *PL*, with variable assignments given by *settings* and the truth value of an well formed formula (wff) given by the *interpretation*.
- that a wff is *valid* if its interpretation is 1 for all possible settings, and is *satisfiable* if there exists a setting such that its interpretation is 1.

Summary
Propositions over sets

We have learned

- about the propositional language $PL(U)$, over propositions over sets modeled by characteristic functions of subsets of U.
- that a *truth set* for a given wff is the set for which the interpretation is a characteristic function.
- that a propositional rule essentially is the application of modus ponens to an implication called the rule.

Summary
Fuzzy Sets and Logic

We have learned

- that *fuzzy sets* are a generalization of crisp sets by relaxing the characteristic function to a *membership function* giving the degree of membership in the set.
- that fuzzy propositions are just like the crisp counterparts,
- and that we can define fuzzy rules just like their crisp counterparts.