Notes for Recitation 21

1 Conditional Expectation and Total Expectation

There are conditional expectations, just as there are conditional probabilities. If R is a random variable and E is an event, then the conditional expectation $\text{Ex}(R \mid E)$ is defined by:

$$\text{Ex}(R \mid E) = \sum_{w \in S} R(w) \cdot \Pr\{w \mid E\}$$

For example, let R be the number that comes up on a roll of a fair die, and let E be the event that the number is even. Let's compute $\text{Ex}(R \mid E)$, the expected value of a die roll, given that the result is even.

$$\begin{aligned}
\text{Ex}(R \mid E) &= \sum_{w \in \{1,\ldots,6\}} R(w) \cdot \Pr\{w \mid E\} \\
&= 1 \cdot 0 + 2 \cdot \frac{1}{3} + 3 \cdot 0 + 4 \cdot \frac{1}{3} + 5 \cdot 0 + 6 \cdot \frac{1}{3} \\
&= 4
\end{aligned}$$

It helps to note that the conditional expectation, $\text{Ex}(R \mid E)$ is simply the expectation of R with respect to the probability measure $\Pr_E()$ defined in PSet 10. So it's linear:

$$\text{Ex}(R_1 + R_2 \mid E) = \text{Ex}(R_1 \mid E) + \text{Ex}(R_2 \mid E).$$

Conditional expectation is really useful for breaking down the calculation of an expectation into cases. The breakdown is justified by an analogue to the Total Probability Theorem:

Theorem 1 (Total Expectation)**.** *Let $E_1, \ldots, E_n$ be events that partition the sample space and all have nonzero probabilities. If R is a random variable, then:*

$$\text{Ex}(R) = \text{Ex}(R \mid E_1) \cdot \Pr\{E_1\} + \cdots + \text{Ex}(R \mid E_n) \cdot \Pr\{E_n\}$$

For example, let R be the number that comes up on a fair die and E be the event that result is even, as before. Then $\overline{E}$ is the event that the result is odd. So the Total Expectation theorem says:

$$\underbrace{\text{Ex}(R)}_{=\,7/2} = \underbrace{\text{Ex}(R \mid E)}_{=\,4} \cdot \underbrace{\Pr\{E\}}_{=\,1/2} + \underbrace{\text{Ex}\left(R \mid \overline{E}\right)}_{=\,?} \cdot \underbrace{\Pr\{E\}}_{=\,1/2}$$

The only quantity here that we don't already know is $\text{Ex}\left(R \mid \overline{E}\right)$, which is the expected die roll, given that the result is odd. Solving this equation for this unknown, we conclude that $\text{Ex}\left(R \mid \overline{E}\right) = 3$.

To prove the Total Expectation Theorem, we begin with a Lemma.

Lemma. *Let R be a random variable, E be an event with positive probability, and I_E be the indicator variable for E. Then*

$$\text{Ex}(R \mid E) = \frac{\text{Ex}(R \cdot I_E)}{\Pr\{E\}} \tag{1}$$

Proof. Note that for any outcome, s, in the sample space,

$$\Pr\{\{s\} \cap E\} = \begin{cases} 0 & \text{if } I_E(s) = 0, \\ \Pr\{s\} & \text{if } I_E(s) = 1, \end{cases}$$

and so

$$\Pr\{\{s\} \cap E\} = I_E(s) \cdot \Pr\{s\}. \tag{2}$$

Now,

$$\begin{aligned}
\text{Ex}(R \mid E) &= \sum_{s \in S} R(s) \cdot \Pr\{\{s\} \mid E\} && (\text{Def of Ex}(\cdot \mid E)) \\
&= \sum_{s \in S} R(s) \cdot \frac{\Pr\{\{s\} \cap E\}}{\Pr\{E\}} && (\text{Def of Pr}\{\cdot \mid E\}) \\
&= \sum_{s \in S} R(s) \cdot \frac{I_E(s) \cdot \Pr\{s\}}{\Pr\{E\}} && (\text{by (2)}) \\
&= \frac{\sum_{s \in S}(R(s) \cdot I_E(s)) \cdot \Pr\{s\}}{\Pr\{E\}} \\
&= \frac{\text{Ex}(R \cdot I_E)}{\Pr\{E\}} && (\text{Def of Ex}(R \cdot I_E))
\end{aligned}$$

□

Now we prove the Total Expectation Theorem:

Proof. Since the E_i's partition the sample space,

$$R = \sum_i R \cdot I_{E_i} \tag{3}$$

for any random variable, R. So

$$\begin{aligned}
\text{Ex}\left(R\right) &= \text{Ex}\left(\sum_i R \cdot I_{E_i}\right) && \text{(by (3))} \\
&= \sum_i \text{Ex}\left(R \cdot I_{E_i}\right) && \text{(linearity of Ex ())} \\
&= \sum_i \text{Ex}\left(R \mid E_i\right) \cdot \Pr\left\{E_i\right\} && \text{(by (1))}
\end{aligned}$$

□

Problem 1. [**points**] Here's yet another fun 6.042 game! You pick a number between 1 and 6. Then you roll three fair, independent dice.

- If your number never comes up, then you lose a dollar.
- If your number comes up once, then you win a dollar.
- If your number comes up twice, then you win two dollars.
- If your number comes up three times, you win *four* dollars!

What is your expected payoff? Is playing this game likely to be profitable for you or not?

Solution. Let the random variable R be the amount of money won or lost by the player in a round. We can compute the expected value of R as follows:

$$\begin{aligned}
\text{Ex}(R) &= -1 \cdot \Pr\{0 \text{ matches}\} + 1 \cdot \Pr\{1 \text{ match}\} + 2 \cdot \Pr\{2 \text{ matches}\} + 4 \cdot \Pr\{3 \text{ matches}\} \\
&= -1 \cdot \left(\frac{5}{6}\right)^3 + 1 \cdot 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + 2 \cdot 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + 4 \cdot \left(\frac{1}{6}\right)^3 \\
&= \frac{-125 + 75 + 30 + 4}{216} \\
&= \frac{-16}{216}
\end{aligned}$$

You can expect to lose 16/216 of a dollar (about 7.4 cents) in every round. This is a horrible game! ■

Problem 2. **[points]** The number of squares that a piece advances in one turn of the game Monopoly is determined as follows:

- Roll two dice, take the sum of the numbers that come up, and advance that number of squares.
- If you roll *doubles* (that is, the same number comes up on both dice), then you roll a second time, take the sum, and advance that number of additional squares.
- If you roll doubles a second time, then you roll a third time, take the sum, and advance that number of additional squares.
- However, as a special case, if you roll doubles a third time, then you go to jail. Regard this as advancing zero squares overall for the turn.

(a) [pts] What is the expected sum of two dice, given that the same number comes up on both?

Solution. There are six equally-probable sums: 2, 4, 6, 8, 10, and 12. Therefore, the expected sum is:

$$\frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 4 + \ldots + \frac{1}{6} \cdot 12 = 7$$

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(b) [pts] What is the expected sum of two dice, given that different numbers come up? (Use your previous answer and the Total Expectation Theorem.)

Solution. Let the random variables D_1 and D_2 be the numbers that come up on the two dice. Let E be the event that they are equal. The Total Expectation Theorem says:

$$\text{Ex}(D_1 + D_2) = \text{Ex}(D_1 + D_2 \mid E) \cdot \Pr\{E\} + \text{Ex}\left(D_2 + D_2 \mid \overline{E}\right) \cdot \Pr\left\{\overline{E}\right\}$$

Two dice are equal with probability $\Pr\{E\} = 1/6$, the expected sum of two independent dice is 7, and we just showed that $\text{Ex}(D_1 + D_2 \mid E) = 7$. Substituting in these quantities and solving the equation, we find:

$$7 = 7 \cdot \frac{1}{6} + \text{Ex}\left(D_2 + D_2 \mid \overline{E}\right) \cdot \frac{5}{6}$$

$$\text{Ex}\left(D_2 + D_2 \mid \overline{E}\right) = 7$$

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(c) [pts] To simplify the analysis, suppose that we always roll the dice three times, but may ignore the second or third rolls if we didn't previously get doubles. Let the random variable X_i be the sum of the dice on the i-th roll, and let E_i be the event that the i-th roll is doubles. Write the expected number of squares a piece advances in these terms.

Solution. From the total expectation formula, we get:

$$\begin{aligned}
\text{Ex}(\text{advance}) = {} & \text{Ex}\left(X_1 \mid \overline{E_1}\right) \cdot \Pr\left\{\overline{E_1}\right\} \\
& + \text{Ex}\left(X_1 + X_2 \mid E_1 \cap \overline{E_2}\right) \cdot \Pr\left\{E_1 \cap \overline{E_2}\right\} \\
& + \text{Ex}\left(X_1 + X_2 + X_3 \mid E_1 \cap E_2 \cap \overline{E_3}\right) \cdot \Pr\left\{E_1 \cap E_2 \cap \overline{E_3}\right\} \\
& + \text{Ex}\left(0 \mid E_1 \cap E_2 \cap E_3\right) \cdot \Pr\left\{E_1 \cap E_2 \cap E_3\right\}
\end{aligned}$$

Then using linearity of (conditional) expectation, we refine this to

$$\begin{aligned}
& \text{Ex}(\text{advance}) \\
& = \text{Ex}\left(X_1 \mid \overline{E_1}\right) \cdot \Pr\left\{\overline{E_1}\right\} \\
& \quad + \left(\text{Ex}\left(X_1 \mid E_1 \cap \overline{E_2}\right) + \text{Ex}\left(X_2 \mid E_1 \cap \overline{E_2}\right)\right) \cdot \Pr\left\{E_1 \cap \overline{E_2}\right\} \\
& \quad + \left(\text{Ex}\left(X_1 \mid E_1 \cap E_2 \cap \overline{E_3}\right) + \text{Ex}\left(X_2 \mid E_1 \cap E_2 \cap \overline{E_3}\right) + \text{Ex}\left(X_3 \mid E_1 \cap E_2 \cap \overline{E_3}\right)\right) \\
& \qquad \cdot \Pr\left\{E_1 \cap E_2 \cap \overline{E_3}\right\} \\
& \quad + 0.
\end{aligned}$$

Using mutual independence of the rolls, we simplify this to

$$\begin{aligned}
& \text{Ex}(\text{advance}) \\
& = \text{Ex}\left(X_1 \mid \overline{E_1}\right) \cdot \Pr\left\{\overline{E_1}\right\} \qquad (4) \\
& \quad + \left(\text{Ex}\left(X_1 \mid E_1\right) + \text{Ex}\left(X_2 \mid \overline{E_2}\right)\right) \cdot \Pr\left\{E_1\right\} \cdot \Pr\left\{\overline{E_2}\right\} \\
& \quad + \left(\text{Ex}\left(X_1 \mid E_1\right) + \text{Ex}\left(X_2 \mid E_2\right) + \text{Ex}\left(X_3 \mid \overline{E_3}\right)\right) \cdot \Pr\left\{E_1\right\} \cdot \Pr\left\{E_2\right\} \cdot \Pr\left\{\overline{E_3}\right\}
\end{aligned}$$

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(d) [pts] What is the expected number of squares that a piece advances in Monopoly?

Solution. We plug the values from parts (a) and (b) into equation (4):

$$\begin{aligned}
\text{Ex}(\text{advance}) &= 7 \cdot \frac{5}{6} + (7+7) \cdot \frac{1}{6} \cdot \frac{5}{6} + (7+7+7) \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \\
&= 8\frac{19}{72}
\end{aligned}$$

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