

## Problems for Recitation 8

### 1 Build-up error

Recall a graph is **connected** iff there is a path between every pair of its vertices.

**False Claim.** *If every vertex in a graph has positive degree, then the graph is connected.*

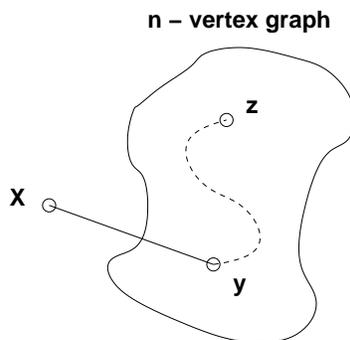
1. Prove that this Claim is indeed false by providing a counterexample.
2. Since the Claim is false, there must be a logical mistake in the following bogus proof. Pinpoint the *first* logical mistake (unjustified step) in the proof.

*Proof.* We prove the Claim above by induction. Let  $P(n)$  be the proposition that if every vertex in an  $n$ -vertex graph has positive degree, then the graph is connected.

**Base cases:** ( $n \leq 2$ ). In a graph with 1 vertex, that vertex cannot have positive degree, so  $P(1)$  holds vacuously.

$P(2)$  holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

**Inductive step:** We must show that  $P(n)$  implies  $P(n + 1)$  for all  $n \geq 2$ . Consider an  $n$ -vertex graph in which every vertex has positive degree. By the assumption  $P(n)$ , this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex  $x$  to obtain an  $(n + 1)$ -vertex graph:



All that remains is to check that there is a path from  $x$  to every other vertex  $z$ . Since  $x$  has positive degree, there is an edge from  $x$  to some other vertex,  $y$ . Thus, we can

obtain a path from  $x$  to  $z$  by going from  $x$  to  $y$  and then following the path from  $y$  to  $z$ . This proves  $P(n + 1)$ .

By the principle of induction,  $P(n)$  is true for all  $n \geq 0$ , which proves the Claim. □

## 2 The Grow Algorithm

Yesterday in lecture, we saw the following algorithm for constructing a minimum-weight spanning tree (MST) from an edge-weighted  $N$ -vertex graph  $G$ .

ALG-GROW:

1. Label the edges of the graph  $e_1, e_2, \dots, e_t$  so that  $wt(e_1) \leq wt(e_2) \dots \leq wt(e_t)$ .
2. Let  $S$  be the empty set.
3. For  $i = 1 \dots t$ , if  $S \cup \{e_i\}$  does not contain a cycle, then extend  $S$  with the edge  $e_i$ .
4. Output  $S$ .

### 2.1 Analysis of ALG-GROW

In this problem you may assume the following lemma from the problem set:

**Lemma 1.** *Suppose that  $T = (V, E)$  is a simple, connected graph. Then  $T$  is a tree iff  $|E| = |V| - 1$ .*

In this exercise you will prove the following theorem.

**Theorem.** *For any connected, weighted graph  $G$ , ALG-GROW produces an MST of  $G$ .*

(a) Prove the following lemma.

**Lemma 2.** *Let  $T = (V, E)$  be a tree and let  $e$  be an edge not in  $E$ . Then,  $G = (V, E \cup \{e\})$  contains a cycle.*

(Hint: Suppose  $G$  does *not* contain a cycle. Is  $G$  a tree?)

(b) Prove the following lemma.

**Lemma 3.** *Let  $T = (V, E)$  be a spanning tree of  $G$  and let  $e$  be an edge not in  $E$ . Then there exists an edge  $e' \neq e$  in  $E$  such that  $T^* = (V, E - \{e'\} \cup \{e\})$  is a spanning tree of  $G$ .*

(Hint: Adding  $e$  to  $E$  introduces a cycle in  $(V, E \cup \{e\})$ .)

(c) Prove the following lemma.

**Lemma 4.** *Let  $T = (V, E)$  be a spanning tree of  $G$ , let  $e$  be an edge not in  $E$  and let  $S \subseteq E$  such that  $S \cup \{e\}$  does not contain a cycle. Then there exists an edge  $e' \neq e$  in  $E - S$  such that  $T^* = (V, E - \{e'\} \cup \{e\})$  is a spanning tree of  $G$ .*

(Hint: Modify your proof to part (b). Of all possible edges  $e' \neq e$  that can be removed to construct  $T^*$ , at least one is not in  $S$ .)

(d) Prove the following lemma.

**Lemma 5.** *Define  $S_m$  to be the set consisting of the first  $m$  edges selected by ALG-GROW from a connected graph  $G$ . Let  $P(m)$  be the predicate that if  $m \leq |V|$  then  $S_m \subseteq E$  for some MST  $T = (V, E)$  of  $G$ . Then  $\forall m . P(m)$ .*

(Hint: Use induction. There are two cases:  $m + 1 > |V|$  and  $m + 1 \leq |V|$ . In the second case, there are two subcases.)

(e) Prove the theorem. (Hint: Lemma 5 says there exists an MST  $T = (V, E)$  for  $G$  such that  $S \subseteq E$ . Use contradiction to rule out the case in which  $S$  is a proper subset of  $E$ .)

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