LECTURE 23

LECTURE OUTLINE

- Interior point methods
- Constrained optimization case - Barrier method
- Conic programming cases
- Linear programming - Path following

BARRIER METHOD

- Inequality constrained problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \qquad g_j(x) \le 0, \quad j = 1, \ldots, r, \end{aligned}$$

where f and g_j are real-valued convex and X is closed convex.

- We assume that the interior (relative to X) set

$$S = \big\{x \in X \mid g_j(x) < 0,\, j = 1, \ldots, r\big\}$$

is nonempty.

- Note that because S is convex, any feasible point can be approached through S (the Line Segment Principle).
- The barrier method is an approximation method.
- It replaces the indicator function of the constraint set

$$\delta\big(x \mid \mathrm{cl}(S)\big)$$

by a smooth approximation within the relative interior of S.

BARRIER FUNCTIONS

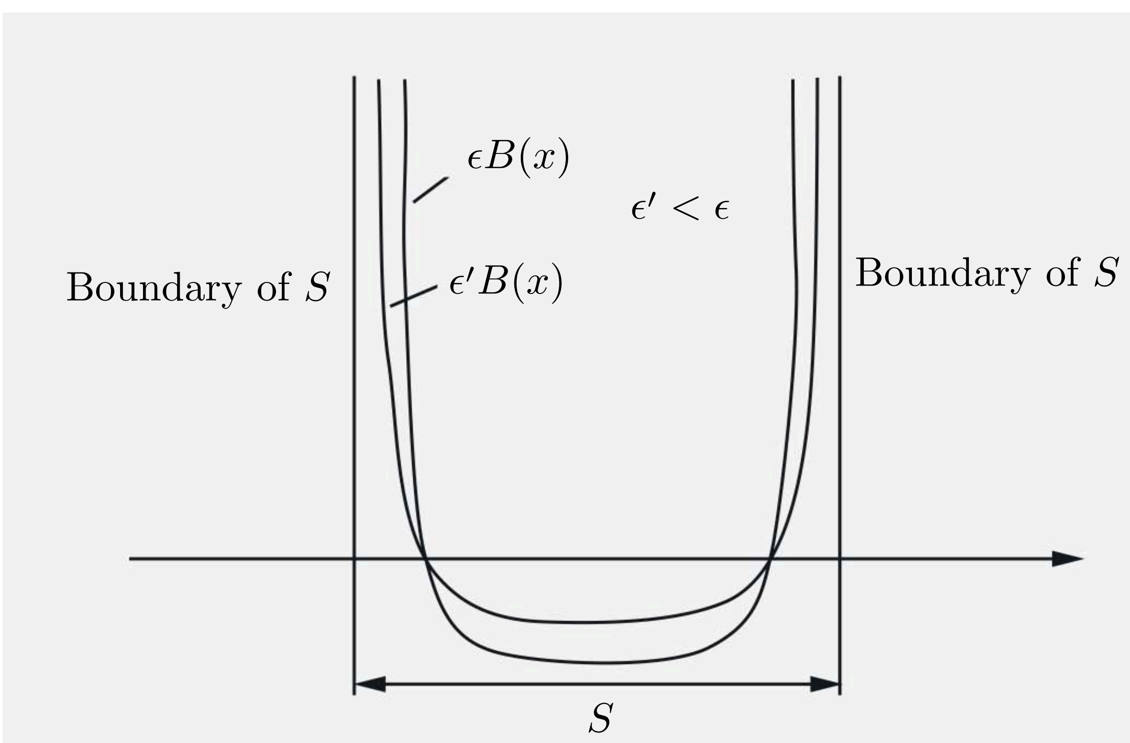
- Consider a *barrier function*, that is continuous and goes to ∞ as any one of the constraints $g_j(x)$ approaches 0 from negative values.
- Examples:

$$B(x) = -\sum_{j=1}^{r} \ln\{-g_j(x)\}, \qquad B(x) = -\sum_{j=1}^{r} \frac{1}{g_j(x)}.$$

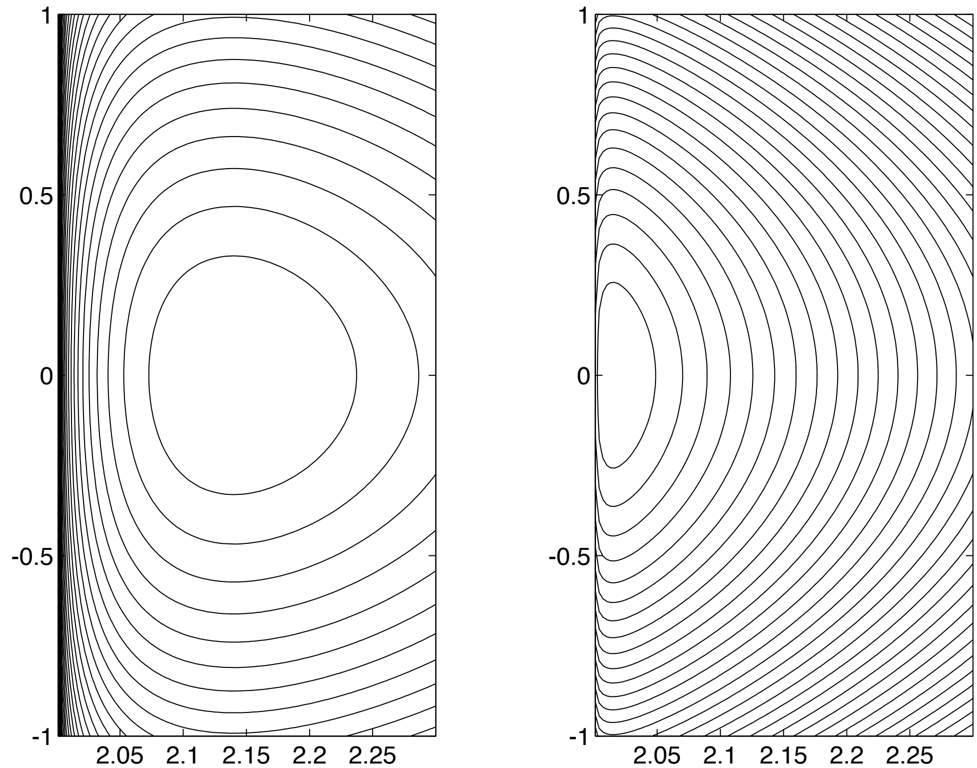
- Barrier method:

$$x^k = \arg\min_{x \in S}\{f(x) + \epsilon_k B(x)\}, \qquad k = 0, 1, \ldots,$$

where the parameter sequence $\{\epsilon_k\}$ satisfies $0 < \epsilon_{k+1} < \epsilon_k$ for all k and $\epsilon_k \to 0$.



BARRIER METHOD - EXAMPLE



$$\begin{aligned} &\text{minimize} \quad f(x) = \tfrac{1}{2}\big((x^1)^2 + (x^2)^2\big) \\ &\text{subject to} \quad 2 \le x^1, \end{aligned}$$

with optimal solution $x^* = (2, 0)$.

- Logarithmic barrier: $B(x) = -\ln(x^1 - 2)$
- We have $x_k = \big(1 + \sqrt{1 + \epsilon_k}\,, 0\big)$ from

$$x_k \in \arg\min_{x^1 > 2} \Big\{ \tfrac{1}{2}\big((x^1)^2 + (x^2)^2\big) - \epsilon_k \ln(x^1 - 2) \Big\}$$

- As ϵ_k is decreased, the unconstrained minimum x_k approaches the constrained minimum $x^* = (2, 0)$.
- As $\epsilon_k \to 0$, computing x_k becomes more difficult because of ill-conditioning (a Newton-like method is essential for solving the approximate problems).

CONVERGENCE

- Every limit point of a sequence $\{x_k\}$ generated by a barrier method is a minimum of the original constrained problem.

Proof: Let $\{x\}$ be the limit of a subsequence $\{x_k\}_{k \in K}$. Since $x_k \in S$ and X is closed, x is feasible for the original problem.

If x is not a minimum, there exists a feasible x^* such that $f(x^*) < f(x)$ and therefore also an interior point $\tilde{x} \in S$ such that $f(\tilde{x}) < f(x)$. By the definition of x_k,

$$f(x_k) + \epsilon_k B(x_k) \leq f(\tilde{x}) + \epsilon_k B(\tilde{x}), \qquad \forall\ k,$$

so by taking limit

$$f(x) + \liminf_{k \to \infty,\, k \in K} \epsilon_k B(x_k) \leq f(\tilde{x}) < f(x)$$

Hence $\liminf_{k \to \infty,\, k \in K} \epsilon_k B(x_k) < 0$.

If $x \in S$, we have $\lim_{k \to \infty,\, k \in K} \epsilon_k B(x_k) = 0$, while if x lies on the boundary of S, we have by assumption $\lim_{k \to \infty,\, k \in K} B(x_k) = \infty$. Thus

$$\liminf_{k \to \infty} \epsilon_k B(x_k) \geq 0,$$

– a contradiction.

SECOND ORDER CONE PROGRAMMING

- Consider the SOCP

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \;\; A_i x - b_i \in C_i, \;\; i = 1, \ldots, m, \end{aligned}$$

where $x \in \Re^n$, c is a vector in $\Re^n$, and for $i = 1, \ldots, m$, A_i is an $n_i \times n$ matrix, b_i is a vector in $\Re^{n_i}$, and C_i is the second order cone of $\Re^{n_i}$.

- We approximate this problem with

$$\begin{aligned} &\text{minimize} \quad c'x + \epsilon_k \sum_{i=1}^{m} B_i(A_i x - b_i) \\ &\text{subject to} \;\; x \in \Re^n, \end{aligned}$$

where B_i is the logarithmic barrier function:

$$B_i(y) = -\ln\left(y_{n_i}^2 - (y_1^2 + \cdots + y_{n_i-1}^2)\right), \quad y \in \text{int}(C_i),$$

and $\{\epsilon_k\}$ is a positive sequence with $\epsilon_k \to 0$.

- Essential to use Newton's method to solve the approximating problems.
- Interesting complexity analysis

SEMIDEFINITE PROGRAMMING

- Consider the dual SDP

$$\begin{aligned} &\text{maximize} \quad b'\lambda \\ &\text{subject to} \quad C - (\lambda_1 A_1 + \cdots + \lambda_m A_m) \in D, \end{aligned}$$

where D is the cone of positive semidefinite matrices.

- The logarithmic barrier method uses approximating problems of the form

$$\text{maximize} \quad b'\lambda + \epsilon_k \ln\Big(\det(C - \lambda_1 A_1 - \cdots - \lambda_m A_m)\Big)$$

over all $\lambda \in \Re^m$ such that $C - (\lambda_1 A_1 + \cdots + \lambda_m A_m)$ is positive definite.

- Here $\epsilon_k > 0$ and $\epsilon_k \to 0$.

- Furthermore, we should use a starting point such that $C - \lambda_1 A_1 - \cdots - \lambda_m A_m$ is positive definite, and Newton's method should ensure that the iterates keep $C - \lambda_1 A_1 - \cdots - \lambda_m A_m$ within the positive definite cone.

LINEAR PROGRAMS/LOGARITHMIC BARRIER

- Apply logarithmic barrier to the linear program

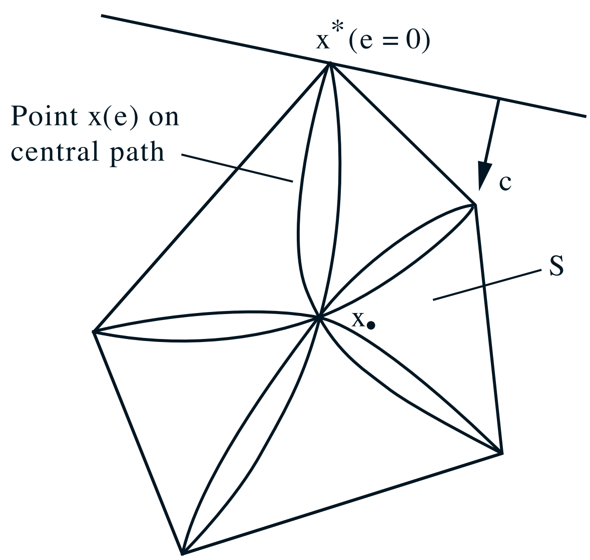
$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad Ax = b, \qquad x \geq 0, \end{aligned} \tag{LP}$$

The method finds for various $\epsilon > 0$,

$$x(\epsilon) = \arg\min_{x \in S} F_\epsilon(x) = \arg\min_{x \in S} \left\{ c'x - \epsilon \sum_{i=1}^n \ln x_i \right\},$$

where $S = \{x \mid Ax = b,\, x > 0\}$. We assume that S is nonempty and bounded.

- As $\epsilon \to 0$, $x(\epsilon)$ follows the *central path*



- All central paths start at the *analytic center*

$$x_\infty = \arg\min_{x \in S} \left\{ -\sum_{i=1}^n \ln x_i \right\},$$

and end at optimal solutions of (LP).

PATH FOLLOWING W/ NEWTON'S METHOD

- Newton's method for minimizing F_ϵ:

$$\tilde{x} = x + \alpha(x - x),$$

where x is the pure Newton iterate

$$x = \arg \min_{Az=b} \left\{ \nabla F_\epsilon(x)'(z - x) + \tfrac{1}{2}(z - x)'\nabla^2 F_\epsilon(x)(z - x) \right\}$$

- By straightforward calculation

$$x = x - Xq(x,\epsilon),$$

$$q(x,\epsilon) = \frac{Xz}{\epsilon} - e, \quad e = (1 \ldots 1)', \quad z = c - A'\lambda,$$

$$\lambda = (AX^2A')^{-1}AX\big(Xc - \epsilon e\big),$$

and X is the diagonal matrix with x_i, $i = 1, \ldots, n$ along the diagonal.

- View $q(x,\epsilon)$ as a "normalized" Newton incement [the Newton increment $(x-x)$ transformed by X^{-1} that maps x into e].

- Consider $\|q(x,\epsilon)\|$ as a *proximity measure* of the current point to the point $x(\epsilon)$ on the central path.

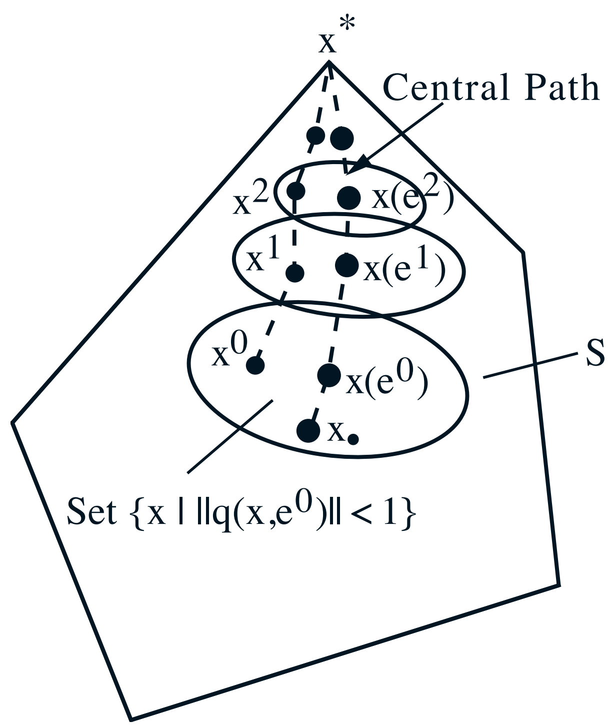
KEY RESULTS

- It is sufficient to minimize F_ϵ approximately, up to where $\|q(x,\epsilon)\| < 1$.

- **Fact 1:** If $x > 0$, $Ax = b$, and $\|q(x,\epsilon)\| < 1$,

$$c'x - \min_{Ay=b,\ y\geq 0} c'y \leq \epsilon\big(n + \sqrt{n}\big).$$

Defines a "tube of convergence".

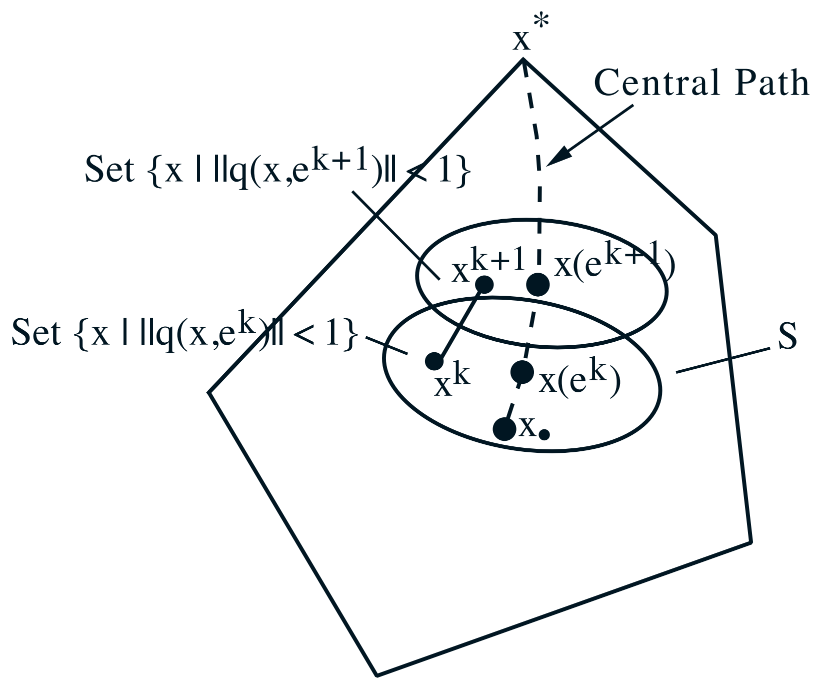


- **Fact 2:** The "termination set" $\big\{x \mid \|q(x,\epsilon)\| < 1\big\}$ is part of the region of quadratic convergence.

- **Fact 2:** If $\|q(x,\epsilon)\| < 1$, then the pure Newton iterate x satisfies

$$\|q(x,\epsilon)\| \leq \|q(x,\epsilon)\|^2 < 1.$$

SHORT STEP METHODS



- **Idea:** Use a **single** Newton step before changing ϵ (a little bit, so the next point stays within the "tube of convergence").

Proposition Let $x > 0$, $Ax = b$, and suppose that for some $\gamma < 1$ we have $\|q(x,\epsilon)\| \leq \gamma$. Then if $\epsilon = (1 - \delta n^{-1/2})\epsilon$ for some $\delta > 0$,

$$\|q(x,\epsilon)\| \leq \frac{\gamma^2 + \delta}{1 - \delta n^{-1/2}}.$$

In particular, if

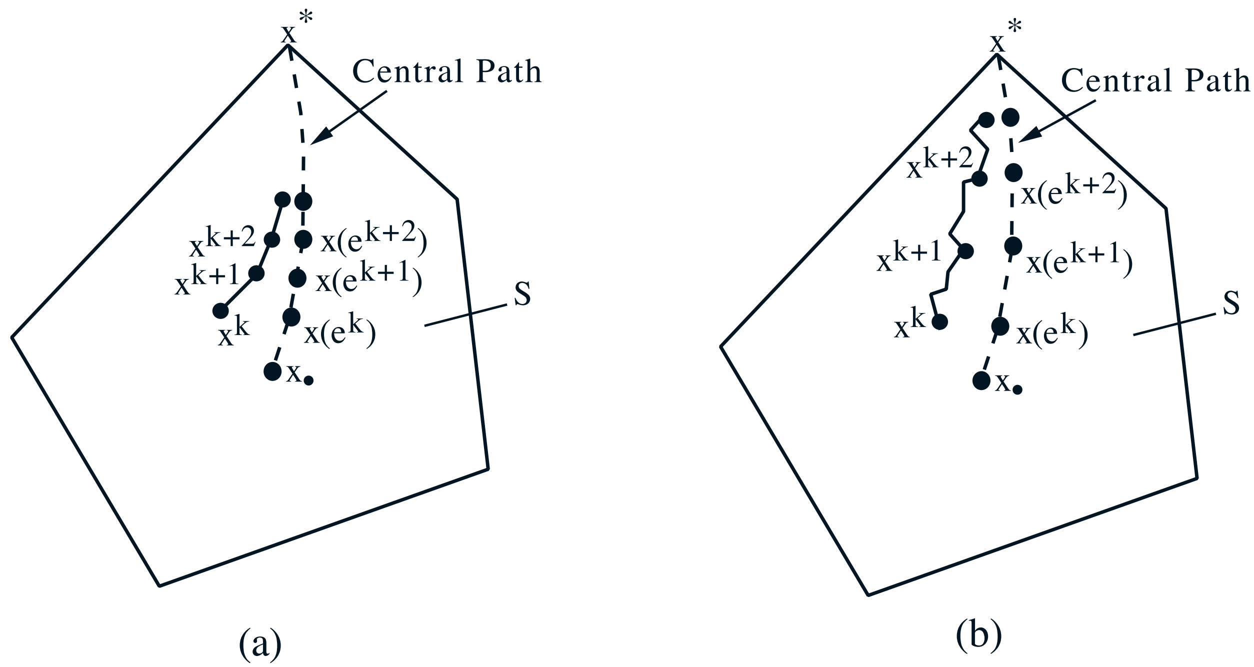
$$\delta \leq \gamma(1-\gamma)(1+\gamma)^{-1},$$

we have $\|q(x,\epsilon)\| \leq \gamma$.

- Can be used to establish nice complexity results; but ϵ must be reduced VERY slowly.

LONG STEP METHODS

- Main features:
  - Decrease ϵ faster than dictated by complexity analysis.
  - Use more than one Newton step per (approximate) minimization.
  - Use line search as in unconstrained Newton's method.
  - Require much smaller number of (approximate) minimizations.



Short Step method Long Step method

- The methodology generalizes to quadratic programming and convex programming.

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