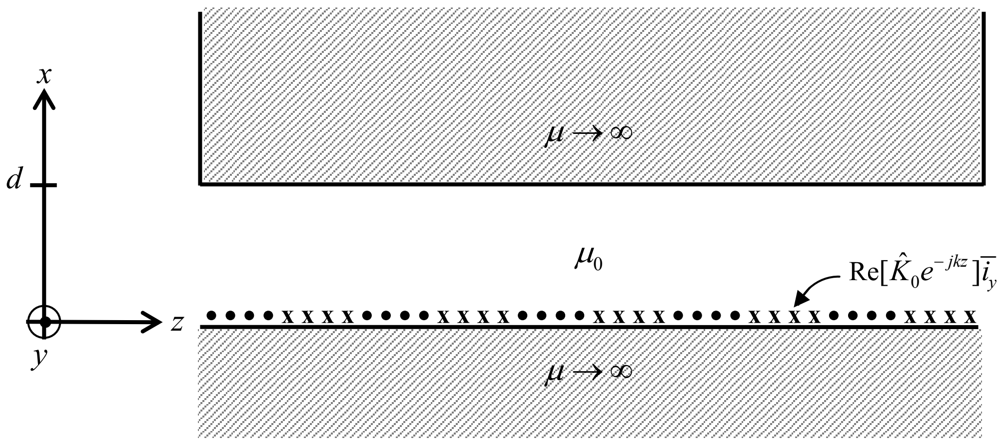
6.642 Continuum Electromechanics
Fall 2008

Continuum Electromechanics
6.642 Mid-Term Exam
November 5, 2008

Formula sheets are located after page 4.

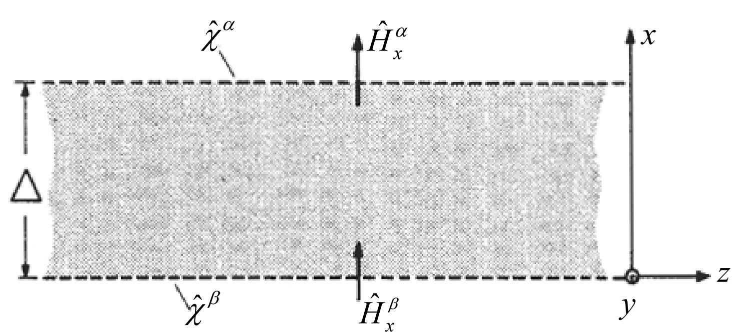
1. (30 points)



A current sheet $\mathrm{Re}[\hat{K}_0 e^{-jkz}]\bar{i}_y$ is placed on the $x = 0$ surface of a material with infinite magnetic permeability ($\mu \to \infty$) for $x < 0$. Another infinite magnetic permeability material extends from $d < x < \infty$.

Free space with magnetic permeability μ_0 extends over the region $0 < x < d$.

The magnetic field (H_x) – magnetic scalar potential (χ) relations for the planar layer below



for variables of the form

$$\chi(x,z) = \mathrm{Re}[\hat{\chi}(x) e^{-jkz}]$$

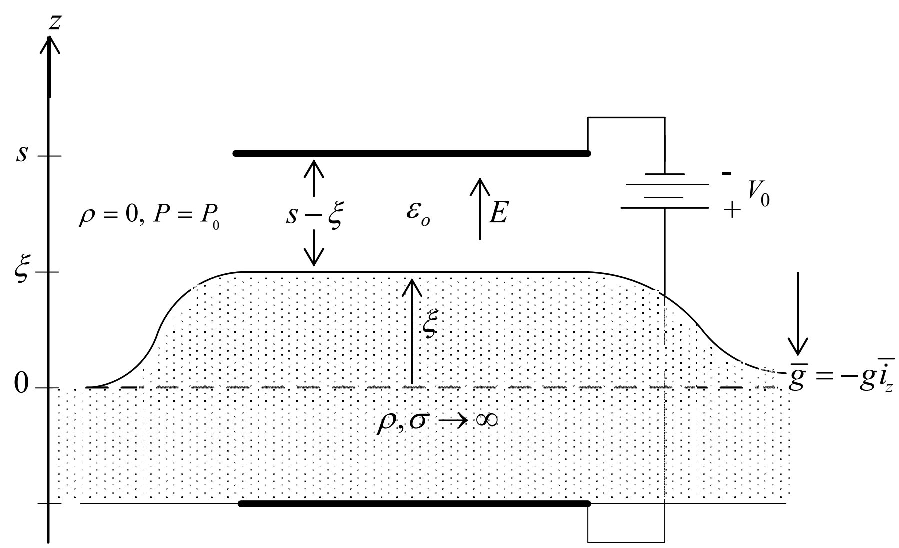
are

$$\begin{bmatrix} \hat{H}_x^\alpha \\ \hat{H}_x^\beta \end{bmatrix} = k \begin{bmatrix} -\coth k\Delta & \dfrac{1}{\sinh k\Delta} \\ -\dfrac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{\chi}^\alpha \\ \hat{\chi}^\beta \end{bmatrix}$$

where $\overline{H}(x,z) = -\nabla\chi(x,z) = \mathrm{Re}[(\hat{H}_x(x)\overline{i}_x + \hat{H}_z(x)\overline{i}_z)e^{-jkz}]$
There is no magnetic field dependence on y.

a) What are the boundary conditions on the magnetic field at the $x = 0_+$ and $x = d_-$ surfaces? What are the values of the magnetic scalar potential $\hat{\chi}(x = 0_+)$ and $\hat{\chi}(x = d_-)$?
b) What are the complex amplitudes of the magnetic field $\overline{H}(x,z)$ at $x = 0_+$ and at $x = d_-$?
c) What is the magnetic force per unit area (on a wave length $2\pi / k$) $\overline{F}$ on the infinite magnetic permeability layer that extends $d < x < \infty$?

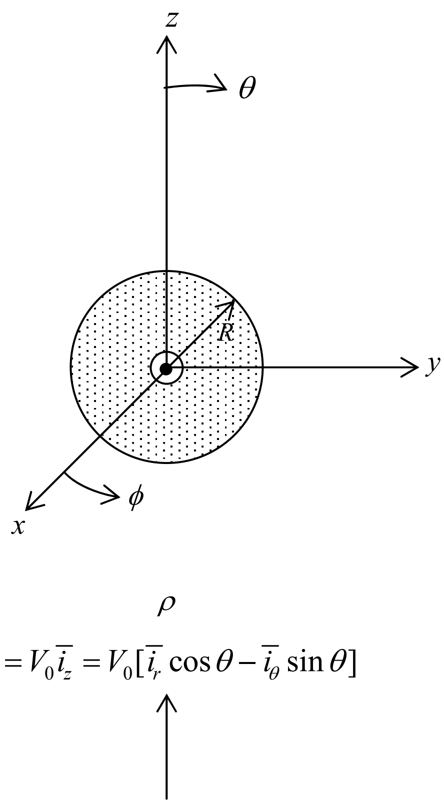
2. (35 points)



A perfectly conducting incompressible liquid $(\sigma \to \infty)$ with mass density ρ partially fills the gap between parallel plate electrodes stressed by voltage V_0. The applied voltage lifts the fluid interface between the parallel plate electrodes by a height ξ where $\xi < s$. The upper electrode in free space is at $z = s$. When the applied voltage is zero the fluid interface is located at $z = 0$. The region outside the liquid is free space with permittivity ε_o, mass density of zero ($\rho = 0$), and atmospheric pressure P_o. The gravitational acceleration is $\overline{g} = -g\overline{i}_z$ and surface tension effects are negligible.

a) What is the electric field for $\xi < z < s$ between the upper electrode at $z = s$ and the perfectly conducting fluid interface at $z = \xi$?
b) What is the fluid pressure $p(\xi_-)$ just below the interface at $z = \xi_-$?
c) Find an expression that relates liquid rise ξ $(\xi < s)$ to voltage V_0 and other given parameters.
d) At what voltage is $\xi = \dfrac{s}{2}$?

3. (35 points)



An inviscid incompressible liquid with mass density ρ has uniform irrotational flow $(\nabla \times \overline{v} = 0)$. The flow at $r = \infty$ is uniform and z directed

$$\overline{v} = V_0\overline{i}_z = V_0[\overline{i}_r \cos\theta - \overline{i}_\theta \sin\theta]$$

The flow is incident on a solid sphere of radius R. The inviscid liquid can flow along the sphere so that $v_\theta(r = R_+) \neq 0$ but cannot penetrate the surface so that $v_r(r = R_+) = 0$. Because the irrotational flow has $\nabla \times \overline{v} = 0$, a velocity scalar potential Φ can be defined, $\overline{v} = -\nabla\Phi$. Because the fluid is also incompressible, $\nabla \bullet \overline{v} = 0$, the velocity scalar potential for $r > R$ obeys Laplace's equation, $\nabla^2\Phi = 0$ where $\Phi(r,\theta)$ does not depend on angle ϕ. The flow does not vary with time and gravity effects are negligible.

a) What are the boundary conditions on the velocity scalar potential at $r = R_+$ and at $r = \infty$?
b) Solve for the velocity scalar potential $\Phi(r,\theta)$.
c) Solve for the velocity field $\overline{v}(r,\theta)$ for $r > R$.
d) What is the magnitude of the velocity $|\overline{v}(r,\theta)|$?
e) If the pressure at $r = R_+$ and $\theta = 0$ is P_0, what is the pressure at $r = R_+$, $\theta = \pi/2$?
f) What is the equation for the velocity streamlines?

Hint: $\int \frac{\cos d\theta}{\sin\theta} = \ln(\sin\theta) + \text{constant}$

$$\int \frac{1 + \frac{R^3}{2r^3}}{r(1 - \frac{R^3}{r^3})}\, dr = \frac{1}{2}\ln[r^2(1 - \frac{R^3}{r^3})] + \text{constant}$$

g) For the velocity streamline that passes through the point $x = 0$, $y = y_0$, $z = 0$ equivalent to $r = y_0$, $\theta = \dfrac{\pi}{2}$, $\phi = \dfrac{\pi}{2}$, for what value of y does the streamline pass through when $x = 0$ and $z = -\infty$, equivalent to $r = \infty$, $\theta = \pi$, $\phi = \dfrac{\pi}{2}$? Find y when $y_0 = R$ and when $y_0 = 2R$.

Hint: $y = r \ \sin\theta \sin\phi$

6.642 Formula Sheet #1

Cartesian Coordinates (x, y, z)

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z$$

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \mathbf{i}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{i}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical Coordinates (r, ϕ, z)

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi + \frac{\partial f}{\partial z}\mathbf{i}_z$$

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \mathbf{i}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{i}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{i}_z\frac{1}{r}\left[\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates (r, θ, ϕ)

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi$$

$$\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial(r^2 A_r)} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times A = \mathbf{i}_r\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right] + \mathbf{i}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial r}\right] + \mathbf{i}_\phi\frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Cartesian		*Cylindrical*		*Spherical*
x	=	$r\cos\phi$	=	$r\sin\theta\cos\phi$
y	=	$r\sin\phi$	=	$r\sin\theta\sin\phi$
z	=	z	=	$r\cos\theta$
$\mathbf{i}_x$	=	$\cos\phi\mathbf{i}_r - \sin\phi\mathbf{i}_\phi$	=	$\sin\theta\cos\phi\mathbf{i}_r + \cos\theta\cos\phi\mathbf{i}_\theta - \sin\phi\mathbf{i}_\phi$
$\mathbf{i}_y$	=	$\sin\phi\mathbf{i}_r + \cos\phi\mathbf{i}_\phi$	=	$\sin\theta\sin\phi\mathbf{i}_r + \cos\theta\sin\phi\mathbf{i}_\theta + \cos\phi\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_\theta$

Cylindrical		*Cartesian*		*Spherical*
r	=	$\sqrt{x^2+y^2}$	=	$r\sin\theta$
ϕ	=	$\tan^{-1}\frac{y}{x}$	=	ϕ
z	=	z	=	$r\cos\theta$
$\mathbf{i}_r$	=	$\cos\phi\mathbf{i}_x + \sin\phi\mathbf{i}_y$	=	$\sin\theta\mathbf{i}_r + \cos\theta\mathbf{i}_\theta$
$\mathbf{i}_\phi$	=	$-\sin\phi\mathbf{i}_x + \cos\phi\mathbf{i}_y$	=	$\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_\theta$

Spherical		*Cartesian*		*Cylindrical*
r	=	$\sqrt{x^2+y^2+z^2}$	=	$\sqrt{r^2+z^2}$
θ	=	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	=	$\cos^{-1}\frac{z}{\sqrt{r^2+z^2}}$
ϕ	=	$\cot^{-1}\frac{x}{y}$	=	ϕ
$\mathbf{i}_r$	=	$\sin\theta\cos\phi\mathbf{i}_x + \sin\theta\sin\phi\mathbf{i}_y + \cos\theta\mathbf{i}_z$	=	$\sin\theta\mathbf{i}_r + \cos\theta\mathbf{i}_z$
$\mathbf{i}_\theta$	=	$\cos\theta\cos\phi\mathbf{i}_x + \cos\theta\sin\phi\mathbf{i}_y - \sin\theta\mathbf{i}_z$	=	$\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_z$
$\mathbf{i}_\phi$	=	$-\sin\phi\mathbf{i}_x + \cos\phi\mathbf{i}_y$	=	$\mathbf{i}_\phi$

Geometric relations between coordinates and unit vectors for Cartesian, cylindrical, and spherical coordinate systems.

6.642 Formula Sheet #1

Vector Identities

$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}$

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$\nabla\cdot(\nabla\times\mathbf{A}) = 0$

$\nabla\times(\nabla f) = 0$

$\nabla(fg) = f\nabla g + g\nabla f$

$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$

$\nabla\cdot(f\mathbf{A}) = f\nabla\cdot\mathbf{A} + (\mathbf{A}\cdot\nabla)f$

$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

$\nabla\times(f\mathbf{A}) = \nabla f\times\mathbf{A} + f\nabla\times\mathbf{A}$

$(\nabla\times\mathbf{A})\times\mathbf{A} = (\mathbf{A}\cdot\nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A}\cdot\mathbf{A})$

$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

Integral Theorems

Line Integral of a Gradient

$$\int_a^b \nabla f\cdot d\mathbf{l} = f(b) - f(a)$$

Divergence Theorem:

$$\int_V \nabla\cdot\mathbf{A}dV = \oint_S \mathbf{A}\cdot d\mathbf{S}$$

Corollaries

$$\int_V \nabla f dV = \oint_S f d\mathbf{S}$$

$$\int_V \nabla\times\mathbf{A}dV = -\oint_S \mathbf{A}\times d\mathbf{S}$$

Stokes' Theorem:

$$\oint_L \mathbf{A}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

Corollary

$$\oint_L f d\mathbf{l} = -\int_S \nabla f\times d\mathbf{S}$$

Maxwell's Equations

Integral	*Differential*	*Boundary Conditions*
Faraday's Law		
$\oint_L \mathbf{E}'\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$	$\mathbf{n}\times(\mathbf{E}_2' - \mathbf{E}_1') = 0$
Ampere's Law with Maxwell's Displacement Current Correction		
$\oint_L \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}_f\cdot d\mathbf{S} + \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{S}$	$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial d}$	$\mathbf{n}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f$
Gauss's Law		
$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho_f dV$	$\nabla\cdot\mathbf{D} = \rho_f$	$\mathbf{n}\cdot(\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f$
$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\mathbf{n}\cdot(\mathbf{B}_2 - \mathbf{B}_1) = 0$
Conservation of Charge		
$\oint_S \mathbf{J}_f\cdot d\mathbf{S} + \frac{d}{dt}\int_V \rho_f dV = 0$	$\nabla\cdot\mathbf{J}_f + \frac{\partial\rho_f}{\partial t} = 0$	$\mathbf{n}\cdot(\mathbf{J}_2 - \mathbf{J}_1) + \frac{\partial\sigma_f}{\partial t} = 0$

Usual Linear Constitutive Laws

$\mathbf{D} = \epsilon\mathbf{E}$

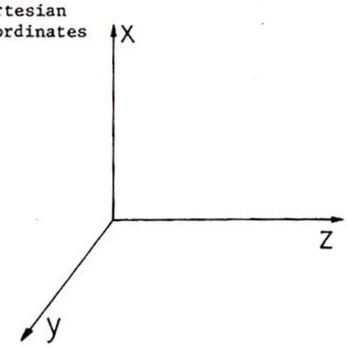
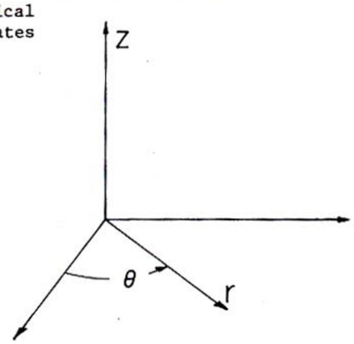
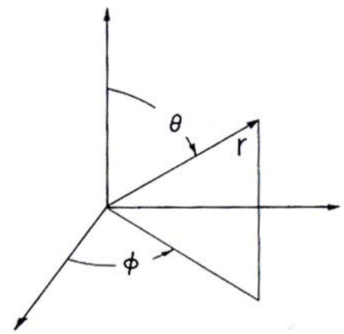
$\mathbf{B} = \mu\mathbf{H}$

$\mathbf{J_f} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \sigma\mathbf{E}'$ [Ohm's law for moving media with velocity $\mathbf{v}$]

Physical Constants

Constant	Symbol	Value	units
Speed of light in vacuum	c	$2.9979\times10^8 \approx 3\times10^8$	m/sec
Elementary electron charge	e	1.602×10^{-19}	coul
Electron rest mass	m_e	9.11×10^{-31}	kg
Electron charge to mass ratio	$\frac{e}{m_e}$	1.76×10^{11}	coul/kg
Proton rest mass	m_p	1.67×10^{-27}	kg
Boltzmann constant	k	1.38×10^{-23}	joule/°K
Gravitation constant	G	6.67×10^{-11}	nt-m^2/(kg)^2
Acceleration of gravity	g	9.807	m/(sec)^2
Permittivity of free space	ϵ_0	$8.854\times10^{-12} \approx \frac{10^{-9}}{36\pi}$	farad/m
Permeability of free space	μ_0	$4\pi\times10^{-7}$	henry/m
Planck's constant	h	6.6256×10^{-34}	joule-sec
Impedance of free space	$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$	$376.73 \approx 120\pi$	ohms
Avogadro's number	N	6.023×10^{23}	atoms/mole

APPENDIX A. Differential Operators in Cartesian, Cylindrical and Spherical Coordinates

Operator	Cartesian coordinates	Cylindrical coordinates	Spherical coordinates
			
$(\nabla\cdot\vec{A})$	$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$	$\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$
$\nabla\Phi$	$\frac{\partial\Phi}{\partial x}\vec{i}_x+\frac{\partial\Phi}{\partial y}\vec{i}_y+\frac{\partial\Phi}{\partial z}\vec{i}_z$	$\frac{\partial\Phi}{\partial r}\vec{i}_r+\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\vec{i}_\theta+\frac{\partial\Phi}{\partial z}\vec{i}_z$	$\frac{\partial\Phi}{\partial r}\vec{i}_r+\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\vec{i}_\theta+\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\vec{i}_\phi$
$(\nabla^2\Phi)$	$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}$	$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\Phi}{\partial r})+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\partial^2\Phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Phi}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$
$(\nabla\times\vec{A})$	$\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\vec{i}_x$ $+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\vec{i}_y$ $+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\vec{i}_z$	$\left(\frac{1}{r}\frac{\partial A_z}{\partial\theta}-\frac{\partial A_\theta}{\partial z}\right)\vec{i}_r$ $+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\vec{i}_\theta$ $+\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)-\frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\vec{i}_z$	$\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right)\vec{i}_r$ $+\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\vec{i}_\theta$ $+\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)-\frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\vec{i}_\phi$

Appendix A in Melcher, James R. *Continuum Electromechanics*. Cambridge, MA: MIT Press, 1981. ISBN: 9780262131650.

$\nabla^2\vec A$	$\left[\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}\right]\vec i_x$ $+\left[\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2}\right]\vec i_y$ $+\left[\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2}\right]\vec i_z$	$\left[\frac{\partial}{\partial r}\left(\frac1r\frac{\partial}{\partial r}(rA_r)\right)+\frac{1}{r^2}\frac{\partial^2A_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta}+\frac{\partial^2A_r}{\partial z^2}\right]\vec i_r$ $+\left[\frac{\partial}{\partial r}\left(\frac1r\frac{\partial}{\partial r}(rA_\theta)\right)+\frac{1}{r^2}\frac{\partial^2A_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial A_r}{\partial\theta}+\frac{\partial^2A_\theta}{\partial z^2}\right]\vec i_\theta$ $+\left[\frac1r\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2A_z}{\partial\theta^2}+\frac{\partial^2A_z}{\partial z^2}\right]\vec i_z$	$\left[\nabla^2A_r-\frac{2A_r}{r^2}-\frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta}-\frac{2A_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right]\vec i_r$ $+\left[\nabla^2A_\theta+\frac{2}{r^2}\frac{\partial A_r}{\partial\theta}-\frac{A_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}\right]\vec i_\theta$ $+\left[\nabla^2A_\phi-\frac{A_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}\right]\vec i_\phi$
$\vec C\cdot\nabla\vec A$	$\left(C_x\frac{\partial A_x}{\partial x}+C_y\frac{\partial A_x}{\partial y}+C_z\frac{\partial A_x}{\partial z}\right)\vec i_x$ $+\left(C_x\frac{\partial A_y}{\partial x}+C_y\frac{\partial A_y}{\partial y}+C_z\frac{\partial A_y}{\partial z}\right)\vec i_y$ $+\left(C_x\frac{\partial A_z}{\partial x}+C_y\frac{\partial A_z}{\partial y}+C_z\frac{\partial A_z}{\partial z}\right)\vec i_z$	$\left(C_r\frac{\partial A_r}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_r}{\partial\theta}+C_z\frac{\partial A_r}{\partial z}-\frac{C_\theta A_\theta}{r}\right)\vec i_r$ $+\left(C_r\frac{\partial A_\theta}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_\theta}{\partial\theta}+C_z\frac{\partial A_\theta}{\partial z}+\frac{C_\theta A_r}{r}\right)\vec i_\theta$ $+\left(C_r\frac{\partial A_z}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_z}{\partial\theta}+C_z\frac{\partial A_z}{\partial z}\right)\vec i_z$	$\left(C_r\frac{\partial A_r}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_r}{\partial\theta}+\frac{C_\phi}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{C_\theta A_\theta}{r}-\frac{C_\phi A_\phi}{r}\right)\vec i_r$ $+\left(C_r\frac{\partial A_\theta}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{C_\phi}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}+\frac{C_\theta A_r}{r}-\frac{C_\phi A_\phi\cot\theta}{r}\right)\vec i_\theta$ $+\left(C_r\frac{\partial A_\phi}{\partial r}+\frac{C_\theta}{r}\frac{\partial A_\phi}{\partial\theta}+\frac{C_\phi}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}+\frac{C_\phi A_r}{r}+\frac{C_\phi A_\theta\cot\theta}{r}\right)\vec i_\phi$
$(\vec{\vec T}:\nabla\vec A)$	$T_{xx}\left(\frac{\partial A_x}{\partial x}\right)+T_{yy}\left(\frac{\partial A_y}{\partial y}\right)+T_{zz}\left(\frac{\partial A_z}{\partial z}\right)$ $+T_{xy}\left(\frac{\partial A_x}{\partial y}+\frac{\partial A_y}{\partial x}\right)$ $+T_{yz}\left(\frac{\partial A_y}{\partial z}+\frac{\partial A_z}{\partial y}\right)+T_{zx}\left(\frac{\partial A_z}{\partial x}+\frac{\partial A_x}{\partial z}\right)$	$T_{rr}\left(\frac{\partial A_r}{\partial r}\right)+T_{\theta\theta}\left(\frac1r\frac{\partial A_\theta}{\partial\theta}+\frac{A_r}{r}\right)+T_{zz}\left(\frac{\partial A_z}{\partial z}\right)$ $+T_{r\theta}\left(r\frac{\partial}{\partial r}\left(\frac{A_\theta}{r}\right)+\frac1r\frac{\partial A_r}{\partial\theta}\right)+T_{\theta z}\left(\frac1r\frac{\partial A_z}{\partial\theta}+\frac{\partial A_\theta}{\partial z}\right)$ $+T_{rz}\left(\frac{\partial A_z}{\partial r}+\frac{\partial A_r}{\partial z}\right)$	$T_{rr}\left(\frac{\partial A_r}{\partial r}\right)+T_{\theta\theta}\left(\frac1r\frac{\partial A_\theta}{\partial\theta}+\frac{A_r}{r}\right)+T_{\phi\phi}\left(\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}+\frac{A_r}{r}+\frac{A_\theta\cot\theta}{r}\right)$ $+T_{r\theta}\left(\frac{\partial A_\theta}{\partial r}+\frac1r\frac{\partial A_r}{\partial\theta}-\frac{A_\theta}{r}\right)+T_{r\phi}\left(\frac{\partial A_\phi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{A_\phi}{r}\right)$ $+T_{\theta\phi}\left(\frac1r\frac{\partial A_\phi}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}-\frac{\cot\theta}{r}A_\phi\right)$
$\nabla\cdot\vec{\vec T}$	$\left(\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}+\frac{\partial T_{xz}}{\partial z}\right)\vec i_x$ $+\left(\frac{\partial T_{yx}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{yz}}{\partial z}\right)\vec i_y$ $+\left(\frac{\partial T_{zx}}{\partial x}+\frac{\partial T_{zy}}{\partial y}+\frac{\partial T_{zz}}{\partial z}\right)\vec i_z$	$\left(\frac1r\frac{\partial}{\partial r}(rT_{rr})+\frac1r\frac{\partial}{\partial\theta}T_{r\theta}-\frac1rT_{\theta\theta}+\frac{\partial T_{rz}}{\partial z}\right)\vec i_r$ $+\left(\frac1r\frac{\partial T_{\theta\theta}}{\partial\theta}+\frac{\partial T_{r\theta}}{\partial r}+\frac2rT_{r\theta}+\frac{\partial T_{\theta z}}{\partial z}\right)\vec i_\theta$ $+\left(\frac1r\frac{\partial}{\partial r}(rT_{zr})+\frac1r\frac{\partial T_{z\theta}}{\partial\theta}+\frac{\partial T_{zz}}{\partial z}\right)\vec i_z$	$\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{rr})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{r\theta}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}\right)\vec i_r$ $+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{\theta\theta}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}+\frac{T_{r\theta}}{r}-\frac{\cot\theta}{r}T_{\phi\phi}\right)\vec i_\theta$ $T_{\theta\phi}$? $+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\phi})+\frac1r\frac{\partial T_{\theta\phi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi}+\frac{T_{r\phi}}{r}+\frac{2\cot\theta}{r}T_{\theta\phi}\right)\vec i_\phi$

Appendix A in Melcher, James R. *Continuum Electromechanics*. Cambridge, MA: MIT Press, 1981. ISBN: 9780262131650.

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6.642 FORMULA SHEET

1. **DIFFERENTIAL OPERATORS IN CYLINDRICAL AND SPHERICAL COORDINATES**

If r, ϕ, and z are circular **cylindrical coordinates** and $\hat{\imath}_r$, $\hat{\imath}_\phi$, and $\hat{\imath}_z$ are unit vectors in the directions of increasing values of the corresponding coordinates,

$$\nabla U = grad\, U = \hat{\imath}_r \frac{\partial U}{\partial r} + \hat{\imath}_\phi \frac{1}{r}\frac{\partial U}{\partial \phi} + \hat{\imath}_z \frac{\partial U}{\partial z}$$

$$\nabla \cdot \vec{A} = div \vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{A} = curl \vec{A} = \hat{\imath}_r \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\imath}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\imath}_z \left(\frac{1}{r}\frac{\partial (rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right)$$

$$\nabla^2 U = div\, grad\, U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

If r, θ, and ϕ are **spherical coordinates** and $\hat{\imath}_r$, $\hat{\imath}_\theta$, and $\hat{\imath}_\phi$ are unit vectors in the directions of increasing values of the corresponding coordinates,

$$\nabla U = grad\, U = \hat{\imath}_r \frac{\partial U}{\partial r} + \hat{\imath}_\theta \frac{1}{r}\frac{\partial U}{\partial \theta} + \hat{\imath}_\phi \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}$$

$$\nabla \cdot \vec{A} = div \vec{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \vec{A} = curl \vec{A} = \hat{\imath}_r \left(\frac{1}{r\sin\theta}\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi}\right) + \hat{\imath}_\theta \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\right) + \hat{\imath}_\phi \left(\frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

$$\nabla^2 U = div\, grad\, U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}$$

2. **SOLUTIONS OF LAPLACE'S EQUATIONS**

A. Rectangular coordinates, two dimensions (independent of z):

$$\Phi = e^{kx}(A_1 \sin ky + A_2 \cos ky) + e^{-kx}(B_1 \sin ky + B_2 \cos ky)$$

(or replace e^{kx} and e^{-kx} by $\sinh kx$ and $\cosh kx$).

$$\Phi = Axy + Bx + Cy + D; (k = 0)$$

B. Cylindrical coordinates, two dimensions (independent of z):

$$\Phi = r^n(A_1 \sin n\phi + A_2 \cos n\phi) + r^{-n}(B_1 \sin n\phi + B_2 \cos n\phi)$$

$$\Phi = \ln\frac{R}{r}(A_1\phi + A_2) + B_1\phi + B_2; (n = 0)$$

C. Spherical coordinates, two dimensions (independent of ϕ):

$$\Phi = Ar\cos\theta + \frac{B}{r^2}\cos\theta + \frac{C}{r} + D$$