6.642 Continuum Electromechanics
Fall 2008

6.642 — Continuum Electromechanics	Fall 2008
Final Exam - Solutions 2008	
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Problem 1

A

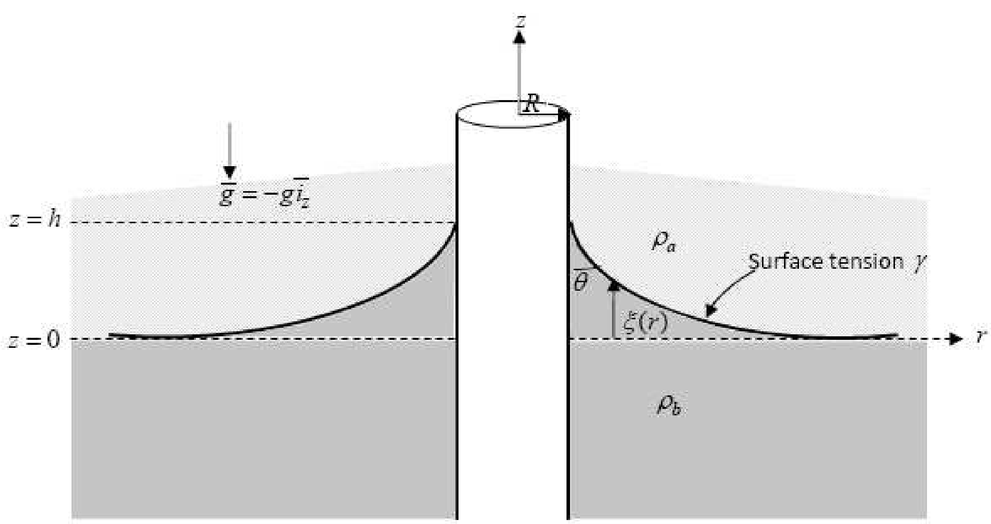

Figure 1: Two superposed fluids surround and wet a cylindrical rod of radius R.

Two superposed fluids surround and wet a cylindrical rod of radius R. The interfacial surface tension is γ and fluid/rod contact angle is θ. The lower fluid has mass density ρ_b and the upper fluid has mass density ρ_a where $\rho_b > \rho_a$. The vertical displacement of the fluid interface $\xi(r)$ is a function of the radial position r rising to a height h at the rod surface at $r = R$. Thus the fluid/rod interface at $r = R$ has the interface height h and contact angle relationships

$\xi\,(r = R) = h,\ \left.\frac{d\xi}{dr}\right|_{r=R} = -\cot\,(\theta)$

We assume that there is no variation with the angle ϕ and that the maximum interfacial displayment h is small enough that a linear analysis for $\xi\,(r)$ can be assumed. Gravity is $\overline{g} = -g\overline{i_z}$.

<u>**Question:**</u> **Far from the cylinder** $(r >> R)$ **the fluid interface is at** $z = 0$. **For** $r = \infty$ **what is the difference in pressures just below and just above the interface,**
$\Delta P\,(r = \infty, z = 0) = P_b\,(r = \infty, z = 0_-) - P_a\,(r = \infty, z = 0_+)$**?**

<u>**Solution:**</u>
Interfacial Force Balance at $r = \infty$, $z = 0$:
$P_b\,(r = \infty, z = 0_-) = P_a\,(r = \infty, z = 0_+)$
$\Delta P\,(r = \infty, z = 0) = P_b\,(r = \infty, z = 0_-) - P_a\,(r = \infty, z = 0_+) = 0$

B

Question: Defining the function $F(r,z) = z - \xi(r)$, the interface between the two fluids is located where $F(r,z) = 0$. To linear terms in $\xi(r)$ what is the unit interfacial normal $\overline{n}$?

Solution:

$F(r,z) = z - \xi(r)$

$$\overline{n} = \left.\frac{\nabla F}{|\nabla F|}\right|_{F(r,z)=0} = \frac{\dfrac{\partial F}{\partial r}\bar{i}_r + \dfrac{1}{r}\dfrac{\partial F}{\partial \phi}\bar{i}_\phi + \dfrac{\partial F}{\partial z}\bar{i}_z}{\left[\left(\dfrac{\partial F}{\partial r}\right)^2 + \left(\dfrac{1}{r}\dfrac{\partial F}{\partial \phi}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2\right]^{1/2}} = \bar{i}_z - \frac{\partial \xi}{\partial r}\bar{i}_r$$

C

Question: The surface tension force per unit area is given by $\overline{T_s} = -\gamma(\nabla \bullet \overline{n})\overline{n}$. What is $\overline{T_s}$?

Solution:

$$\overline{T_s} = -\gamma(\nabla \bullet \overline{n})\overline{n} = -\gamma\left[\frac{1}{r}\frac{\partial(rn_r)}{\partial r} + \frac{1}{r}\frac{\partial n_\phi}{\partial \phi} + \frac{\partial n_z}{\partial z}\right]\overline{n}$$

$n_r = -\frac{\partial \xi}{\partial r}$, $n_\phi = 0$, $n_z = 1$

$$\overline{T}_s = +\gamma\left[\frac{1}{r}\frac{\partial\left(r\frac{\partial \xi}{\partial r}\right)}{\partial r}\right]\left(\bar{i}_z - \frac{\partial \xi}{\partial r}\bar{i}_r\right)$$

D

Question: Using Bernoulli's law and interfacial force balance the governing linear equation for interfacial shape $\xi(r)$ can be written in the form

$A(r)\frac{d^2\xi(r)}{dr^2} + B(r)\frac{d\xi(r)}{dr} + C(r,\xi(r)) = 0$

What are $A(r)$, $B(r)$ and $C(r)$?

Solution:

$P_b(r, z = \xi_-(r)) + \rho_b g\xi(r) = P_b(r = \infty, z = 0_-)$

$P_a(r, z = \xi_+(r)) + \rho_a g\xi(r) = P_a(r = \infty, z = 0_+)$

$P_b(r, z = \xi_-(r)) - P_a(r = \xi_+(r)) - \gamma\nabla \bullet \overline{n} = 0$

$$-g(\rho_b - \rho_a)\xi(r) + \frac{\gamma}{r}\frac{d\left(r\dfrac{d\xi}{dr}\right)}{dr} = 0$$

$$\frac{d^2\xi(r)}{dr^2} + \frac{1}{r}\frac{d\xi(r)}{dr} - \frac{g(\rho_b - \rho_a)\xi(r)}{\gamma} = 0$$

$$r^2\frac{d^2\xi(r)}{dr^2} + r\frac{d\xi(r)}{dr} - \frac{g(\rho_b - \rho_a)}{\gamma}r^2\xi(r) = 0$$

$$A(r)\frac{d^2\xi(r)}{dr^2} + B(r)\frac{d\xi(r)}{dr} - \frac{g(\rho_b - \rho_a)}{\gamma}r^2\xi(r) = 0$$

$$A(r) = r^2, B(r) = r, C(r,\xi(r)) = -\frac{g(\rho_b - \rho_a)}{\gamma}r^2\xi(r)$$

E

Question: Taking $\xi(r = R) = h$ and $\xi(r = \infty) = 0$, solve for $\xi(r)$.

Solution:

$$\xi(r) = C_1 I_0(\alpha r) + C_2 K_0(\alpha r), \alpha = \sqrt{\frac{g(\rho_b - \rho_a)}{\gamma}}$$

$$\xi(r = \infty) = 0 \Rightarrow C_1 = 0$$

$$\xi(r = R) = C_2 K_0(\alpha R) = h \Rightarrow C_2 = \frac{h}{K_0(\alpha R)}$$

$$\xi(r) = h\frac{K_0(\alpha r)}{K_0(\alpha R)} = h\frac{K_0\left(\sqrt{\frac{g(\rho_b - \rho_a)}{\gamma}}r\right)}{K_0\left(\sqrt{\frac{g(\rho_b - \rho_a)}{\gamma}}R\right)}$$

F

Question: How is h related to the contact angle θ?

Solution:

$$\left.\frac{d\xi}{dr}\right|_{r=R} = -\cot\theta = -h\sqrt{\frac{g(\rho_b - \rho_a)}{\gamma}}\left.\frac{K_1\left(\sqrt{\frac{g(\rho_b - \rho_a)}{\gamma}}r\right)}{K_0\left(\sqrt{\frac{g(\rho_b - \rho_a)}{\gamma}}R\right)}\right|_{r=R}$$

$$h = \frac{\cot\theta}{\sqrt{\frac{g(\rho_b - \rho_a)}{\gamma}}}\frac{K_0\left(\sqrt{\frac{g(\rho_b - \rho_a)}{\gamma}}R\right)}{K_1\left(\sqrt{\frac{g(\rho_b - \rho_a)}{\gamma}}R\right)}$$

Problem 2

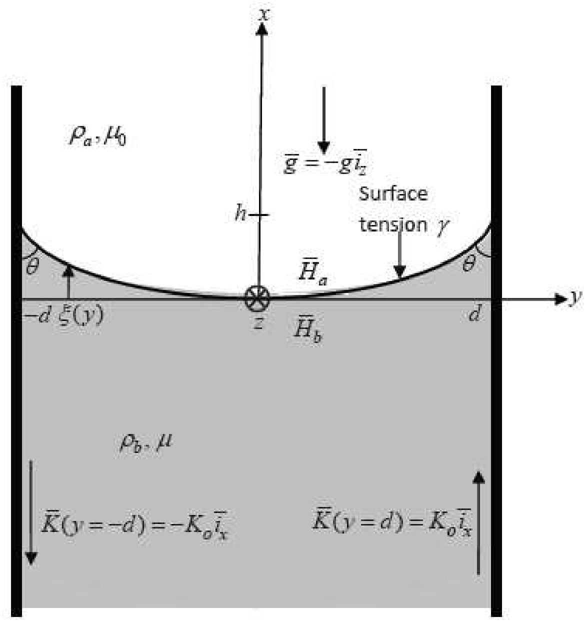

Figure 2: Two superposed and perfectly electrically insulating fluids are contained between parallel plate walls at $y = \pm d$ carrying surface currents $\pm K_0 \overline{i_x}$.

Two superposed and perfectly electrically insulating fluids are contained between vertical plane walls at $y = \pm d$. The fluid interface has surface tension γ and the identical wall contact angles at $y = \pm d$ are θ. The lower fluid is a ferrofluid with mass density ρ_b and magnetic permeability μ and the upper fluid is non-magnetic with mass density ρ_a and magnetic permeability μ_0 with $\rho_b > \rho_a$. The vertical displacement of the fluid interface $\xi(y)$ is a function of position y rising to a height h at $y = \pm d$. Thus the fluid/wall interface at $y = \pm d$ has the interface height h and contact angle relationships

$$\xi(y = d) = \xi(y = -d) = h$$

$$\left.\frac{d\xi}{dy}\right|_{y=d} = -\left.\frac{d\xi}{dy}\right|_{y=-d} = \cot(\theta)$$

The vertical plane walls at $y = \pm d$ are perfectly conducting and carry oppositely directed surface currents $\overline{K}(y = d) = -\overline{K}(y = -d) = K_0 \overline{i_x}$

We assume that there is no variation with the z coordinate and that the maximum interfacial displacement h is small enough that a linear analysis for $\xi(y)$ can be assumed. Gravity is $\overline{g} = -g\overline{i_x}$.

A

Question: The magnetic field is assumed to be spatially uniform in both fluids given by

$$\overline{H} = \begin{cases} \overline{H}_a & (upper\,fluid) \\ \overline{H}_b & (lower\,fluid) \end{cases}$$

What are $\overline{H}_a$ and $\overline{H}_b$ (magnitude and direction)?

<u>Solution:</u>
$\overline{H}_a = \overline{H}_b = -K_0\overline{i}_z$

B

<u>Question:</u> Defining the function $F(x, y) = x - \xi(y)$, the interface between the two fluids is located where $F(x, y) = 0$. To linear terms in $\xi(y)$ what is the interfacial normal $\overline{n}$?

<u>Solution:</u>
$F(x, y) = x - \xi(y)$
$\overline{n} = \nabla F = \overline{i_x} - \frac{\partial \xi}{\partial y}\overline{i_y}$

C

<u>Question:</u> The surface tension force per unit area is given by $\overline{T}_s = -\gamma(\nabla \bullet \overline{n})\overline{n}$. What is $\overline{T}_s$?

<u>Solution:</u>
$\overline{T}_s = -\gamma(\nabla \bullet \overline{n})\overline{n} = -\gamma\overline{n}\left(-\frac{\partial^2 \xi}{\partial y^2}\right) = \gamma\frac{\partial^2 \xi}{\partial y^2}\overline{n}$

D

<u>Question:</u> Using Bernoulli's law within each region find the difference in the pressures just below and above the interface at any position $\xi(y)$,
$\Delta p(y) = P_b(\xi_-(y)) - P_a(\xi_+(y))$
in terms of given parameters and the pressures just below and just above the interface at $y = 0$
$\Delta p(y = 0) = P_b(x = 0_-, y = 0) - P_a(x = 0_+, y = 0)$
Note: It is not yet possible to find the pressure difference $\Delta p(y = 0)$. You will be able to find this in part(f).

<u>Solution:</u>

$$\begin{aligned} P_b(y) + \rho_b g\xi &= P_b(x = 0_-, y = 0) \\ P_a(y) + \rho_a g\xi &= P_a(x = 0_+, y = 0) \end{aligned} \quad \Rightarrow \Delta p(y) = P_b(y) - P_a(y) = \Delta p(y = 0) - g(\rho_b - \rho_a)\xi$$

E

<u>Question:</u> Using the result of part (D) and interfacial force balance including the magnetic surface force the governing linear equation for $\xi(y)$ can be written in the form

$$\frac{d^2\xi(y)}{dy^2} - A\xi(y) = -B$$

What are A and B?

<u>Solution:</u> $$\left(\underbrace{P_b(y) - P_a(y)}_{\Delta p(y)} + \gamma\frac{\partial^2 \xi}{\partial y^2}\right) n_i + \|T_{ij}n_j\| = 0$$

Take $i = x \Rightarrow n_x = 1, n_y = -\frac{\partial \xi}{\partial y}, n_z = 0$

$$T_{ij}n_j = T_{xx}n_x + T_{xy}n_y + \cancelto{0}{T_{xz}n_z}$$
$$T_{xx} = \frac{\mu}{2}\left(\cancelto{0}{H_x^2} - \cancelto{0}{H_y^2} - H_z^2\right) = -\frac{\mu}{2}K_0^2; \|T_{xx}n_x\| = -\frac{(\mu_0-\mu)}{2}K_0^2$$
$$T_{xy} = \mu H_x H_y = 0$$
$$\Delta p_y + \gamma\frac{d^2\xi}{dy^2} + \frac{\mu-\mu_0}{2}K_0^2 = 0$$
$$\gamma\frac{d^2\xi}{dy^2} - g(\rho_b-\rho_a)\xi + \Delta p(y=0) + \frac{(\mu-\mu_0)}{2}K_0^2 = 0$$
$$\frac{d^2\xi}{dy^2} - \frac{(\rho_b-\rho_a)}{\gamma}\xi + \frac{1}{\gamma}\left[\Delta p(y=0) + \frac{(\mu-\mu_0)}{2}K_0^2\right] = 0$$
$$A = \frac{g(\rho_b-\rho_a)}{\gamma}, B = \frac{1}{\gamma}\left[\Delta p(y=0) + \frac{(\mu-\mu_0)}{2}K_0^2\right]$$
$$\frac{d^2\xi}{dy^2} - A\xi = -B$$

F

<u>Question:</u> Taking $\xi(y=d) = \xi(y=-d) = h$ and that $\xi(y=0) = 0$ solve for $\xi(y)$ in terms of given parameters and $\Delta p(y=0)$.

<u>Solution:</u>

$$\xi(y) = \frac{B}{A} + C_1\cosh\sqrt{A}y + C_2\sinh\sqrt{A}y$$
$$\xi(y=d) = h = \frac{B}{A} + C_1\cosh\sqrt{A}d + C_2\sinh\sqrt{A}d$$
$$\xi(y=-d) = h = \frac{B}{A} + C_1\cosh\sqrt{A}d - C_2\sinh\sqrt{A}d$$

Substract:

$$2C_2\sinh\sqrt{A}d = 0 \Rightarrow C_2 = 0$$
$$h = \frac{B}{A} + C_1\cosh\sqrt{A}d \Rightarrow C_1 = \frac{h-\frac{B}{A}}{\cosh\sqrt{A}d}$$
$$\xi(y) = \frac{B}{A} + \frac{h-\frac{B}{A}}{\cosh\sqrt{A}d}\cosh\sqrt{A}y$$

G

<u>Question:</u> Solve for the pressure difference just below and just above the interface at $y=0$, $\Delta p(y=0)$.

<u>Solution:</u>

$$\xi(y=0) = \frac{B}{A} + \frac{h-\frac{B}{A}}{\cosh\sqrt{A}d} = 0 \Rightarrow h = \frac{B}{A}\left(1-\cosh\sqrt{A}d\right)$$
$$B = \frac{Ah}{1-\cosh\sqrt{A}d} = \frac{1}{\gamma}\left[\left(\frac{\mu-\mu_0}{2}\right)K_0^2 + \underbrace{P_b(x=0_-, y=0) - P_a(x=0_+, y=0)}_{\Delta p(y=0)}\right]$$
$$\Delta p(y=0) = P_b(x=0_-, y=0) - P_a(x=0_+, y=0) = -\left(\frac{\mu-\mu_0}{2}\right)K_0^2 - \frac{Ah\gamma}{\cosh\sqrt{A}d - 1}$$

H

Question: How is h related to the contact angle θ?

Solution:

$$\left.\frac{d\xi}{dy}\right|_{y=d} = \cot\theta = \frac{\sqrt{A}\left(h - \frac{B}{A}\right)}{\cosh\sqrt{A}d}\sinh\sqrt{A}d = \sqrt{A}\left(h - \frac{B}{A}\right)\tanh\sqrt{A}d$$

$$\begin{aligned}\frac{B}{A} &= \frac{1}{\gamma A}\left[\Delta p\,(y=0) + \frac{(\mu-\mu_0)}{2}K_0^2\right] \\ &= \frac{1}{\gamma A}\left[\cancel{-(\mu-\mu_0)\frac{K_0^2}{2}} + \frac{Ah\gamma}{\cosh\sqrt{A}d - 1} + \cancel{\left(\frac{\mu-\mu_0}{2}\right)K_0^2}\right] \\ &= \frac{h}{\cosh\sqrt{A}d - 1}\end{aligned}$$

$$\cot\theta = \sqrt{A}h\left(1 - \frac{1}{\cosh\sqrt{A}d - 1}\right)\tanh\sqrt{A}d$$

Problem 3

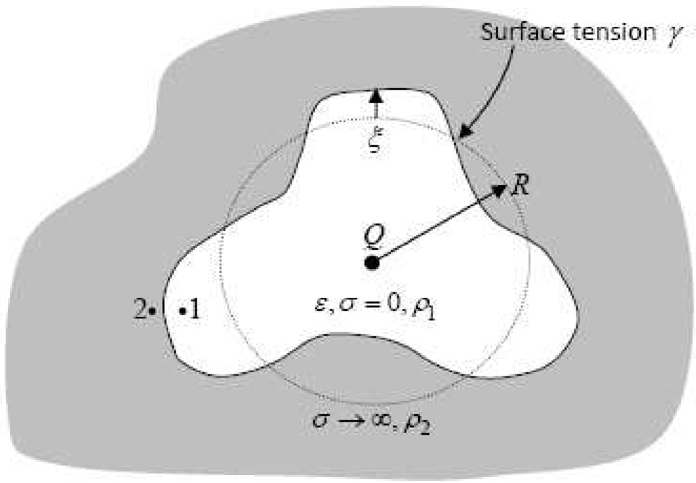

Figure 3: A point charge Q is located at the center of a perfectly insulating liquid spherical drop surrounded by a perfectly conducting liquid of infinite extent.

A point charge Q is located at the center of a perfectly insulating liquid spherical drop with mass density ρ_1 with dielectric permittivity ϵ. This drop is surrounded by a perfectly conducting liquid of mass density ρ_2 that extends to $r = \infty$. The point charge Q is fixed to $r = 0$ and cannot move from this position. The fluid interface has surface tension γ. As the interface is radially perturbed by displacement $\xi(\theta, \phi, t) = Re\left[\hat{\xi} P_n^m(\cos\theta) e^{j(\omega t - m\phi)}\right]$ all perturbation variables change as:

Fluid velocity: $\overline{v}(r, \theta, \phi, t) = Re\left[\left(\hat{v}_r(r)\,\overline{i}_r + \hat{v}_\theta(r)\,\overline{i}_\theta + \hat{v}_\phi(r)\,\overline{i}_\phi\right) P_n^m(\cos\theta) e^{j(\omega t - m\phi)}\right]$ $\quad 0 < r < \infty$ (both regions)

Pressure: $p(r, \theta, \phi, t) = Re\left[\hat{p}(r) P_n^m(\cos\theta) e^{j(\omega t - m\phi)}\right]$ $\quad 0 < r < \infty$ (both regions)

Electric field: $\overline{e}(r, \theta, \phi, t) = Re\left[\left(\hat{e}_r(r)\,\overline{i}_r + \hat{e}_\theta(r)\,\overline{i}_\theta + \hat{e}_\phi(r)\,\overline{i}_\phi\right) P_n^m(\cos\theta) e^{j(\omega t - m\phi)}\right]$ $\quad 0 < r < R + \xi$ (inner droplet)

Electric potential: $\overline{e} = -\nabla\Phi,\ \Phi(r, \theta, \phi, t) = Re\left[\hat{\Phi}(r) P_n^m(\cos\theta) e^{j(wt - m\phi)}\right]$ $\quad 0 < r < R + \xi$ (inner droplet)

A position just inside the interface at $r = (R + \xi)_-$ is labeled 1 and just outside the interface at $r = (R + \xi)_+$ is labeled 2.

A

<u>Question:</u> **What is the equilibrium electric potential $\Phi(r)$, and electric field $\overline{E}(r) = -\nabla\Phi$ within the inner droplet for $0 < r < R$.**

<u>**Solution:**</u>

$$\Phi(r) = \frac{Q}{4\pi\epsilon r},\ E_r(r) = \frac{Q}{4\pi\epsilon r^2} \qquad 0 < r < R$$

B

<u>Question:</u> **What is the equilibrium jump in pressure across the spherical interface $\Delta p(r = R) =$**

$p_1\left(r=R_-\right)-p_2\left(r=R_+\right)$**?**

Solution: Equilibrium $\left(\xi\left(r=R\right)=0\right): p_1-p_2-\frac{2\gamma}{R}-T_{rr1}=0$

$$T_{rr1}=\frac{\epsilon}{2}\left(E_r^2-\cancelto{0}{E_\theta^2}-\cancelto{0}{E_\phi^2}\right)\Bigg|_{r=R}=\frac{\epsilon}{2}\left[E_r\left(r=R\right)\right]^2=\left(\frac{Q}{4\pi R^2}\right)^2\frac{1}{2\epsilon}$$

$$\Delta p=p_1-p_2=\frac{2\gamma}{R}+\frac{1}{2\epsilon}\left(\frac{Q}{4\pi R^2}\right)^2$$

C

Question: What boundary condition must the total electric field satisfy at the $r=R+\xi$ interface? Apply this boundary condition to determine the perturbation electric scalar potential complex amplitude $\hat{\Phi}\left(r=R_-\right)$ in terms of interfacial displacement complex amplitude $\hat{\xi}$.

Solution:

$$\overline{n}\times\overline{E}\left(r=R+\xi\right)=0\Rightarrow$$

$$\left[\bar{i}_r-\frac{1}{R}\frac{\partial\xi}{\partial\theta}\bar{i}_\theta-\frac{1}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\bar{i}_\phi\right]\times\left[\left(E_r\left(r=R\right)+\frac{dE_r}{dr}\bigg|_{r=R}\xi+e_{r1}\right)\bar{i}_r+e_{\theta1}\bar{i}_\theta+e_{\phi1}\bar{i}_\phi\right]_{r=R}\xi=0$$

$$\bar{i}_\phi\left(e_{\theta1}+\frac{1}{R}\frac{\partial\xi}{\partial\theta}E_r\left(r=R\right)\right)-\bar{i}_\theta\left(e_{\phi1}+\frac{E_r\left(r=R\right)}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\right)=0$$

$$\overline{e}=-\nabla\Phi'\Rightarrow e_\theta=-\frac{1}{r}\left.\frac{\partial\Phi'}{\partial\theta}\right|_{r=R}=-\frac{1}{R}\left.\frac{\partial\Phi'}{\partial\theta}\right|_{r=R}$$

$$e_\phi=-\frac{1}{r\sin\theta}\left.\frac{\partial\Phi'}{\partial\phi}\right|_{r=R}=-\frac{1}{R\sin\theta}\left.\frac{\partial\Phi'}{\partial\phi}\right|_{r=R}$$

$$e_{\theta1}+\frac{1}{R}\frac{\partial\xi}{\partial\theta}\frac{Q}{4\pi\epsilon R^2}=-\frac{1}{R}\frac{\partial\left[\Phi'\left(r=R\right)-\frac{Q\xi}{4\pi\epsilon R^2}\right]}{\partial\theta}=0$$

$$\Phi'\left(r=R\right)=\frac{Q\xi}{4\pi\epsilon R^2}$$

$$e_{\phi1}+\frac{E_r\left(r=R\right)}{R\sin\theta}\frac{\partial\xi}{\partial\phi}=-\frac{1}{R\sin\theta}\frac{\partial\left[\Phi'\left(r=R\right)-\frac{Q\xi}{4\pi\epsilon R^2}\right]}{\partial\phi}=0$$

$$\Phi'\left(r=R\right)=\frac{Q\xi}{4\pi\epsilon R^2}$$

$$\hat{\Phi}\left(r=R\right)=\frac{Q\hat{\xi}}{4\pi\epsilon R^2}$$

D

Question: What are the perturbation pressure complex amplitudes $\hat{p}_1$ and $\hat{p}_2$ at both sides of the $r=R+\xi$ interface in terms of interfacial displacement complex amplitude $\hat{\xi}$.

Solution:

$$\hat{p_1}=j\omega\rho_1\left[F_n\left(0,R\right)\hat{\nu}_{r1}+G_n\left(R,0\right)\hat{\nu}_r\left(r=0\right)\right]$$

$\hat{p_2} = j\omega\rho_2 \left[G_n (R, \infty) \hat{\nu}_r (r = \infty) + F_n (\infty, R) \hat{\nu}_{r2}\right]$
$\hat{\nu}_{r1} = \hat{\nu}_{r2} = j\omega\hat{\xi}$

$$F_n [x, y] = \frac{\frac{y}{x}\left[\frac{1}{n}\left(\frac{y}{x}\right)^n + \frac{1}{n+1}\left(\frac{x}{y}\right)^{n+1}\right]}{\left[\frac{1}{y}\left(\frac{x}{y}\right)^n - \frac{1}{x}\left(\frac{y}{x}\right)^n\right]}$$

$$G_n [x, y] = \frac{y}{x}\frac{2n+1}{n(n+1)}\frac{1}{\left[\frac{1}{y}\left(\frac{x}{y}\right)^n - \frac{1}{x}\left(\frac{y}{x}\right)^n\right]}$$

$F_n [0, R] = -\frac{R}{n}, F_n [\infty, R] = \frac{R}{n+1}$

$G_n [R, 0] = 0, G_n [R, \infty] = 0 \, (n \neq 0, 1)$

$\hat{p_1} = j\omega\rho_1 F_n (0, R) \left(j\omega\hat{\xi}\right) = +\frac{\rho_1\omega^2 R}{n}\hat{\xi}$
$\hat{p_2} = j\omega\rho_2 F_n (\infty, R) \left(j\omega\hat{\xi}\right) = -\frac{\rho_2\omega^2 R}{n+1}\hat{\xi}$

E

Question: What is the radial component of the perturbation interfacial stress complex amplitude $\hat{T}_{sr}$ due to surface tension in terms of interfacial displacement complex amplitude $\hat{\xi}$?

Solution:
$\hat{T}_{sr} = -\frac{\gamma}{R^2}(n-1)(n+2)\hat{\xi}$

F

Question: What is the perturbation radial electric field complex amplitude $\hat{e}_r (r = R_-)$ in terms of $\hat{\Phi} (r = R_-)$? Using the results of part (C) express $\hat{e}_r (r = R_-)$ in terms of $\hat{\xi}$.

Solution:
$\hat{e}_{r1} = -\frac{n}{R}\hat{\Phi}_1 = -\frac{nQ\hat{\xi}}{4\pi\epsilon R^3}$

G

Question: Find the dispersion relation. Is the spherical droplet stabilized or destabilized by the electric field from the point charge Q?

Solution:
$\hat{p}_1 - \hat{p}_2 - \frac{\gamma}{R^2}(n-1)(n+2)\hat{\xi} - T_{rj}n_j = 0$
$T_{rj}n_j = T_{rr}n_r + T_{r\theta}n_\theta + T_{r\phi}n_\phi, n_r = 1, n_\theta = -\frac{1}{R}\frac{\partial\xi}{\partial\theta}, n_\phi = -\frac{1}{R\sin\theta}\frac{\partial\xi}{\partial\phi}$

$$T_{rr}|_{r=R+\xi} = \frac{\epsilon}{2}\left[\left(E_r\left(r=R+\xi\right)+e_r\left(r=R\right)\right)^2 - e^2\left(r=R\right) - e_\phi^2\left(r=R\right)\right]$$
$$= \frac{\epsilon}{2}\left[\left(E_r\left(r=R\right)+\left.\frac{dE_r}{dr}\right|_{r=R}\xi + e_r\left(r=R\right)\right)^2\right]$$

$$T_{rr}\left(r=R+\xi\right) = \frac{\epsilon}{2}\left[E_r^2\left(r=R\right)+2E_r\left(r=R\right)\left[\left.\frac{dE_r}{dr}\right|_{r=R}\xi+e_r\left(r=R\right)\right]\right]$$

$$T'_{rr}\left(r=R+\xi\right) = \epsilon E_r\left(r=R\right)\left[\left.\frac{dE_r}{dr}\right|_{r=R}\xi+e_r\left(r=R\right)\right]$$
$$= \not{\epsilon}\frac{Q}{4\pi\not{\epsilon}R^2}\left[-\frac{Q}{2\pi\epsilon R^3}\xi - \frac{n}{R}\Phi'_1\right]$$
$$= -\frac{Q^2}{8\pi^2\epsilon R^5}\xi - \frac{Q}{4\pi R^3}n\frac{Q}{4\pi\epsilon R^2}\xi$$
$$= -\frac{Q^2}{8\pi^2\epsilon R^5}\xi\left[1+\frac{n}{2}\right]$$

$$T_{r\theta}n_\theta|_{r=R+\xi} = \epsilon E_r\left(r=R+\xi\right)e_\theta\left(r=R\right)\left(-\frac{1}{R}\frac{\partial\xi}{\partial\theta}\right)$$
$$= 0 \qquad \text{(Second Order)}$$

$$T_{r\phi}n_\phi|_{r=R+\xi} = \epsilon E_r\left(r=R+\xi\right)e_\phi\left(r=R\right)\left(-\frac{1}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\right)$$
$$= 0 \qquad \text{(Second Order)}$$

$$\hat{p}_1 - \hat{p}_2 - \frac{\gamma}{R^2}\left(n-1\right)\left(n+2\right)\hat{\xi} + \frac{Q^2\hat{\xi}}{8\pi^2\epsilon R^5}\left(1+\frac{n}{2}\right) = 0$$
$$\omega^2 R\left(\frac{\rho_1}{n}+\frac{\rho_2}{n+1}\right) = \frac{\gamma}{R^2}\left(n-1\right)\left(n+2\right) - \frac{Q^2}{8\pi^2\epsilon R^5}\left(1+\frac{n}{2}\right)$$

Electric field destabilizes interface.

H

Question: If (G) is stabilizing, what is the lowest oscillation frequency? If (G) is destabilizing, what is the lowest value of n that is unstable and what is the growth rate of the instability? What value of Q will only have one unstable mode?

Solution:

$n=1$ First unstable mode. $n=2$ and larger are stable if $\frac{4\gamma}{R^2} > \frac{Q^2}{8\pi^2\epsilon R^5}(2)$ or $Q^2 < 16\gamma R^3\pi^2\epsilon$

$|Q| < 4\pi\left[\epsilon\gamma R^3\right]^{1/2}$